

Large-amplitude pairing collective dynamics associated with spontaneously broken global U(1) symmetry

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Outline

Introduction

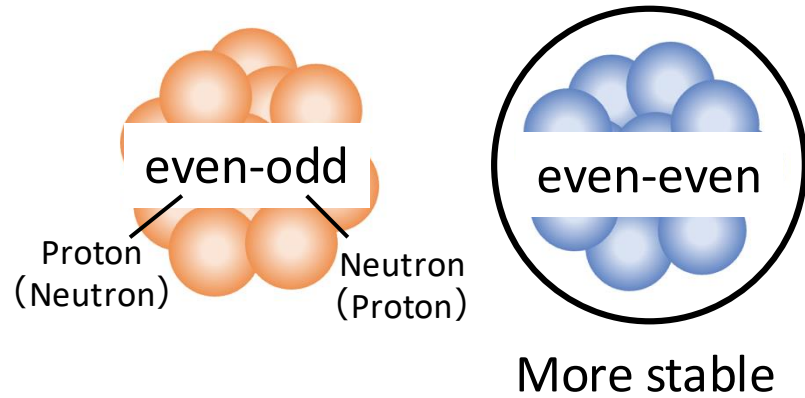
Pairing interaction in nuclei
Pairing collective motion
Critical point
Pairing collective Hamiltonian
Purpose of this study

Method

Results

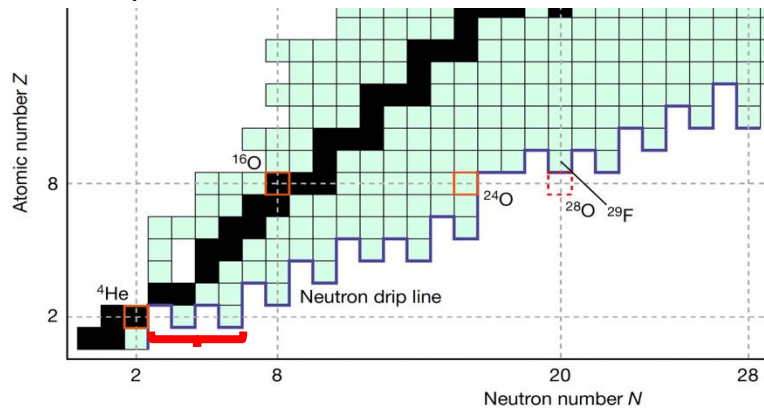
Summary

■ Pairing interaction



■ Phenomena influenced by the pairing interaction

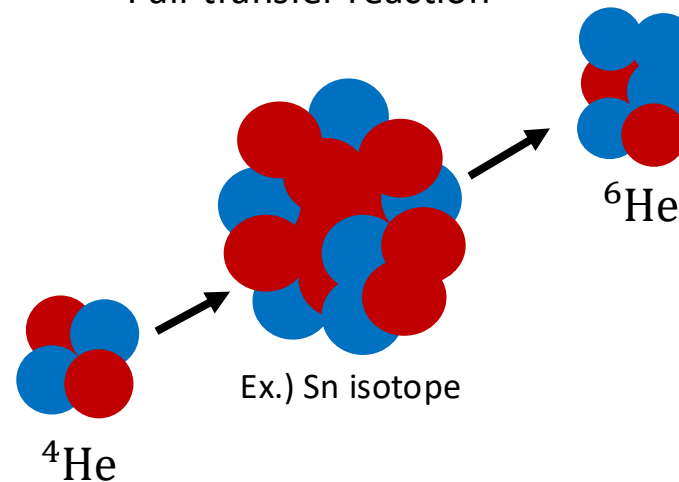
• Drip line



Kouno et al. Nature **620**, p.965–970 (2023)

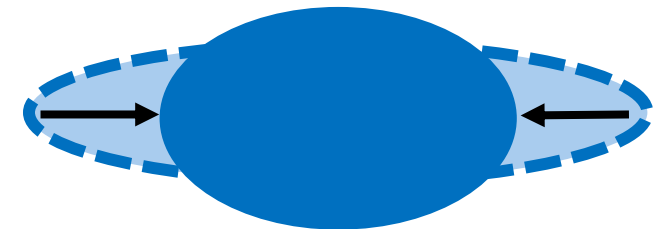
The drip line has a jagged shape.

• Pair transfer reaction



Two-neutron pair addition

• Deformation of nuclei



Reducing the deformation

- Ground-state energies of even-even nuclei from the viewpoint of the pairing

$$E_0(N_0 + \Delta N) = E_0(N_0) + \lambda_n(N_0)\Delta N + E_{\text{pair}}(N_0, \Delta N)$$

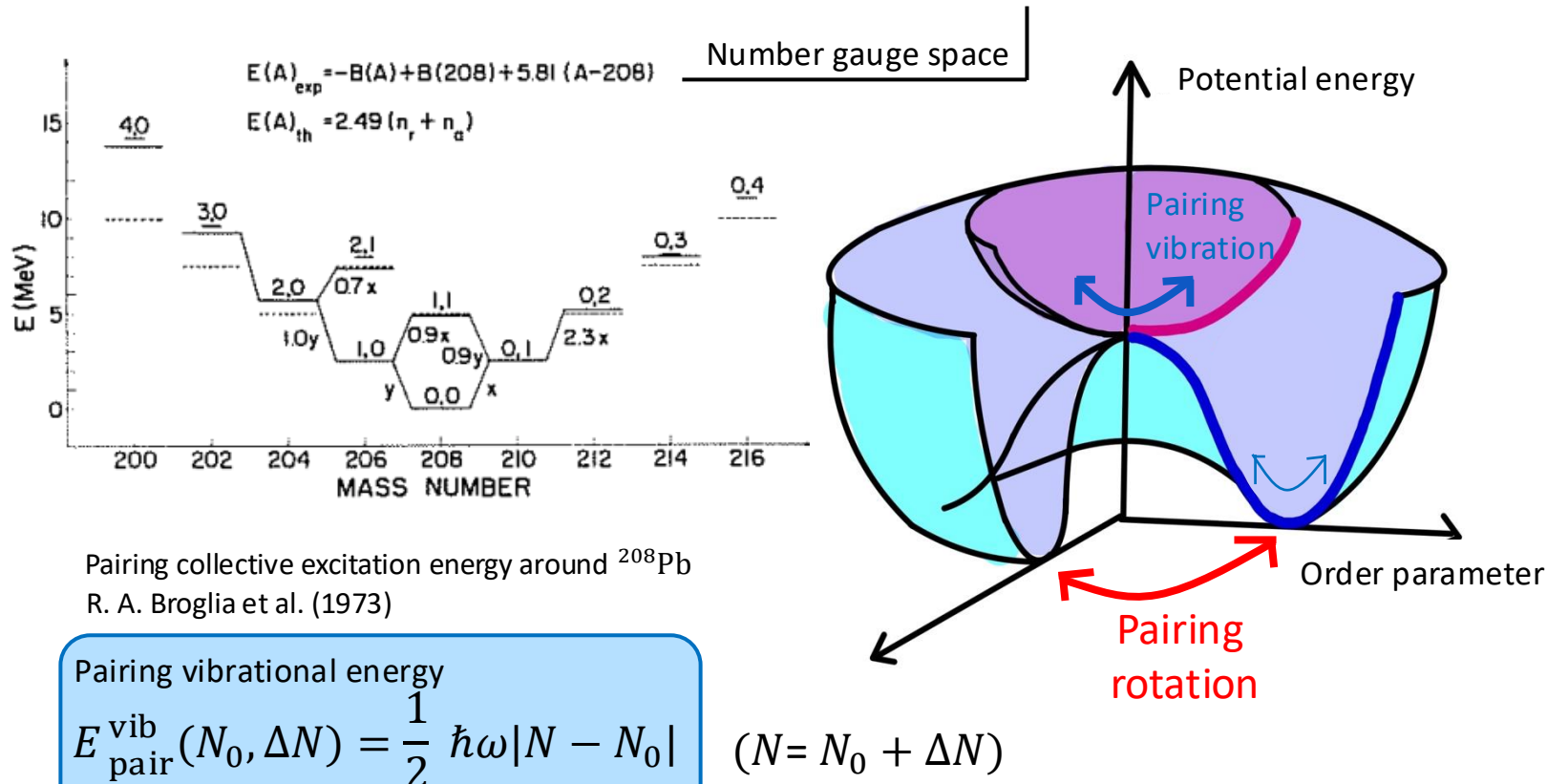
Ground-state energy

Fermi energy

Pair collective excitation energy

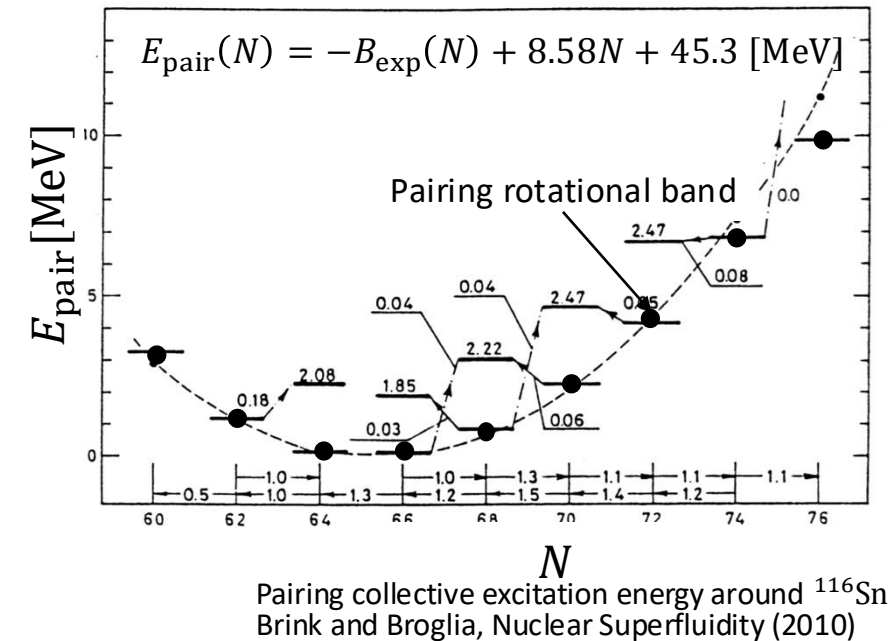
occurs in nuclei near magic numbers (normal phase)

Pairing vibration



Pairing rotation

occurs in open-shell nuclei (superfluid phase)

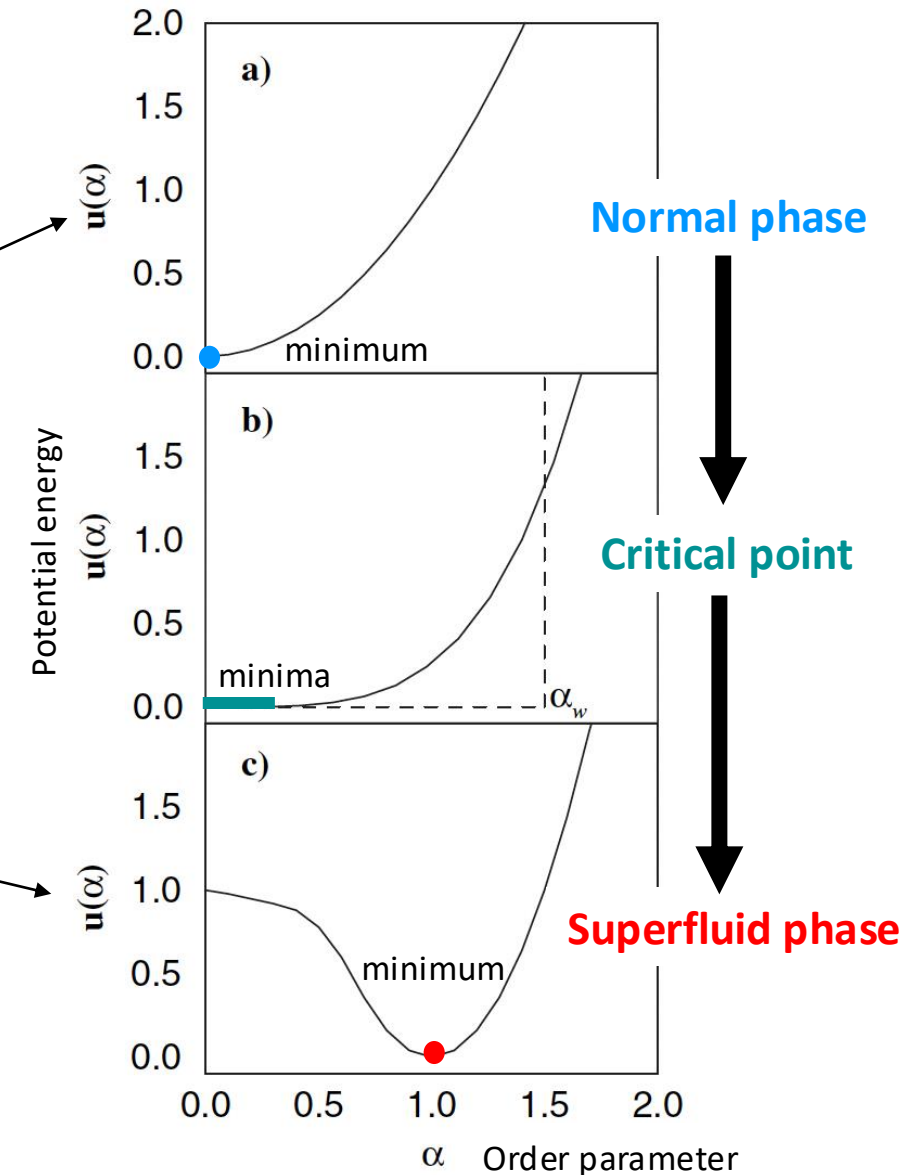
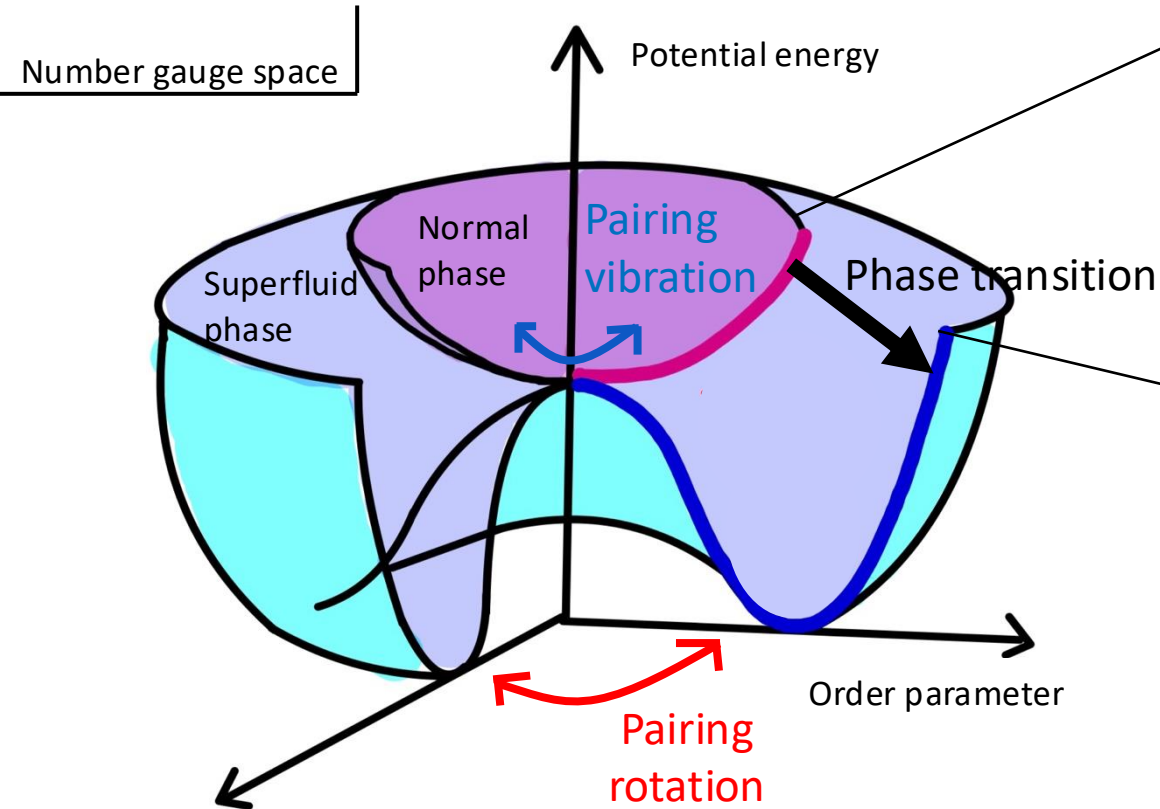


Pairing rotational energy

$$E_{\text{pair}}^{\text{rot}}(N_0, \Delta N) = \frac{(N - N_0)^2}{2\mathcal{I}}$$

Critical point

- The critical point of the pairing transition from normal to superfluid phase
- The doubly magic nucleus ^{208}Pb can be very close to the critical point.
(Clark and Macchiavelli, PRC **77**, 057301 (2008))
- Clark et al. (2008) and others have discussed critical points phenomenologically, but no microscopic analysis has been performed so far.



- To analyze various situations including the critical point, it is necessary to construct the pairing collective Hamiltonian that can describe the pairing rotation and vibration within the unified framework.

Pairing collective Hamiltonian in D. R. Bés et al. (1970)

$$\begin{aligned}\hat{H}_{\text{coll}} &= \hat{V}(\alpha) + \hat{T}_{\text{rot}}(\alpha) + \hat{T}_{\text{vib}}(\alpha) \\ &= -\frac{1}{2\mathcal{M}(\alpha)} \frac{\partial^2}{\partial \alpha^2} - \frac{1}{4} \left(\frac{1}{\mathcal{J}(\alpha)\mathcal{M}(\alpha)} \frac{\partial \mathcal{J}(\alpha)}{\partial \alpha} - \frac{1}{\mathcal{M}(\alpha)^2} \frac{\partial \mathcal{M}(\alpha)}{\partial \alpha} \right) \frac{\partial}{\partial \alpha} + \left(V(\alpha) + \lambda(N - N_0) + \frac{(N - N_0)^2}{2\mathcal{J}(\alpha)} \right)\end{aligned}$$

- Bés et al. (1970) used cranking masses that do not include the effect of the residual interaction, while Clark et al. (2008) used constant masses.

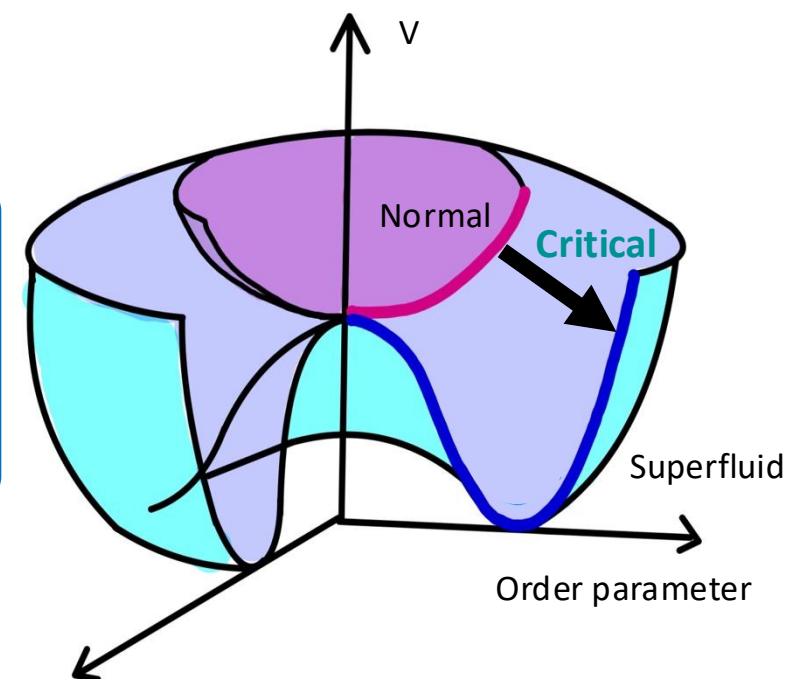
→ We use dynamical collective masses obtained from local QRPA calculations, which take into account the effect of the residual interaction.

Goal of this study

- We construct a **pairing collective Hamiltonian (PCH)** based on nuclear density functional theory capable of the pairing collective dynamics across the entire nuclear chart including the critical point of the phase transition.
- By identifying critical-point nuclei, we **reexamine the magicity** from the perspective of the pairing collective dynamics.

In this talk...

We construct the PCH using the constrained BCS + local QRPA method. We compare the resulting physical quantities with the experimental data and analyze the effect of the residual interaction.



Outline

Introduction

Method

Constrained BCS calculation
local QRPA calculation

Results

Summary

We performed the constrained BCS calculations to obtain the state including the **neutron 1S_0 pairing** in **Pb** isotopes.

We constrain the Hamiltonian with the particle number N and the pairing order parameter α .

$$H'' = \sum_{\mu} e_{\mu} (c_{\mu}^{\dagger} c_{\mu} + c_{\bar{\mu}}^{\dagger} c_{\bar{\mu}}) - \frac{G}{2} \sum_{\mu\nu} (c_{\mu}^{\dagger} c_{\bar{\mu}}^{\dagger} c_{\nu} c_{\bar{\nu}} + c_{\bar{\mu}} c_{\mu} c_{\nu}^{\dagger} c_{\bar{\nu}}^{\dagger})$$

$$= H' + \mu(\hat{\alpha} - \alpha_0) + C(\hat{\alpha} - \alpha_0)^2 \quad \begin{array}{l} \text{Linear constraint} \\ \text{Augmented Lagrangian Method} \\ \text{(A. Staszczak et al. (2010))} \end{array}$$

Pair operator ($\alpha = \langle \hat{\alpha} \rangle$): $\hat{\alpha} = \frac{1}{2} \sum_{\mu} (c_{\mu}^{\dagger} c_{\bar{\mu}}^{\dagger} + c_{\bar{\mu}} c_{\mu})$

The single particle energy e_{μ} is determined by solving the Schrödinger equation using modified oscillator potential (HO + l^2 term + $l \cdot s$ term) .

We solve the variational equation using the BCS wavefunctions.

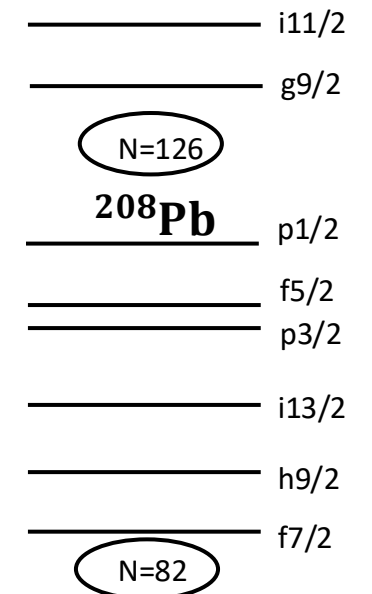
$$\Delta_{\text{eff}} = G \langle \alpha \rangle - \frac{\mu}{2} - C(\langle \alpha \rangle - \alpha_0)$$

Here, we determine the pairing strength **G** so as to reproduce the experimental **pairing gap Δ of ^{188}Pb** .

Then we obtain the collective potential $V(\alpha)$.

$$V(\alpha) = \langle \text{BCS}(\alpha) | H' | \text{BCS}(\alpha) \rangle$$

Model space



Local QRPA : calculates the collective excitation in each collective coordinate using the constrained mean-field (BCS) state.
(→large-amplitude dynamics)

Since the collective potential is soft, we use the local QRPA.

QRPA equation in each collective coordinate

$$[H' + (\mu(\hat{\alpha} - \alpha_0) + C(\hat{\alpha} - \alpha_0)^2)_{\text{MF}}, \Gamma_{\lambda}^{\dagger}] = \omega_{\lambda} \Gamma_{\lambda}^{\dagger} \quad \text{QRPA phonon operator}$$

From this equation, we can write the above equation using A and B matrices, and then diagonalize the QRPA equation.

$$(A + B)(A - B)Q_{\lambda} = \omega_{\lambda}^2 Q_{\lambda} \quad -i(A + B)P_{\lambda} = \omega_{\lambda}^2 Q_{\lambda} \quad \text{We normalize the } P, Q \text{ vectors.}$$

Thouless-Valatin pairing moment of inertia (**P-Mol**, **$J(\alpha)$**)

$$J_{\text{TV}}(\alpha) = 2 \sum_{\mu, \nu > 0} N_{\mu\bar{\mu}}^{20}(\alpha) (A(\alpha) + B(\alpha))_{\mu\bar{\mu}\nu\bar{\nu}}^{-1} N_{\nu\bar{\nu}}^{20}(\alpha)$$

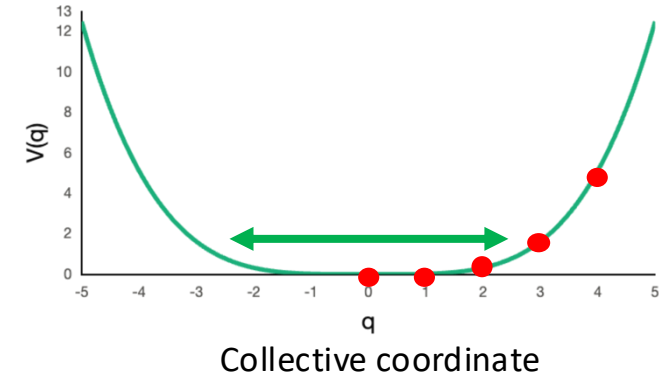
Pairing vibrational mass (**$\mathcal{M}(\alpha)$**)

$$\mathcal{M}(\alpha) = \min \mathcal{M}^{\lambda}(\alpha) = \min \left(\langle \text{BCS}(\alpha) | \left[\hat{\alpha}, \frac{1}{i} \hat{P}^{\lambda}(\alpha) \right] | \text{BCS}(\alpha) \rangle \right)^{-2}$$

Washiyama, Hinohara, and Nakatsukasa, PRC **103**, 014306 (2021)

We insert $V(\alpha)$, **$J(\alpha)$** , and **$\mathcal{M}(\alpha)$** of ^{208}Pb into the PCH.

$$\hat{H}_{\text{coll}} = -\frac{1}{2\mathcal{M}(\alpha)} \frac{\partial^2}{\partial \alpha^2} - \frac{1}{4} \left(\frac{1}{J(\alpha)\mathcal{M}(\alpha)} \frac{\partial J(\alpha)}{\partial \alpha} - \frac{1}{\mathcal{M}(\alpha)^2} \frac{\partial \mathcal{M}(\alpha)}{\partial \alpha} \right) \frac{\partial}{\partial \alpha} + (V(\alpha) + \lambda(N - N_0) + \frac{(N - N_0)^2}{2J(\alpha)})$$



Outline

Introduction

Method

Results

Collective potential $V(\alpha)$, Mass $\mathcal{M}(\alpha)$, P-Mol $\mathcal{I}(\alpha)$

Mixing between rotational and vibrational modes

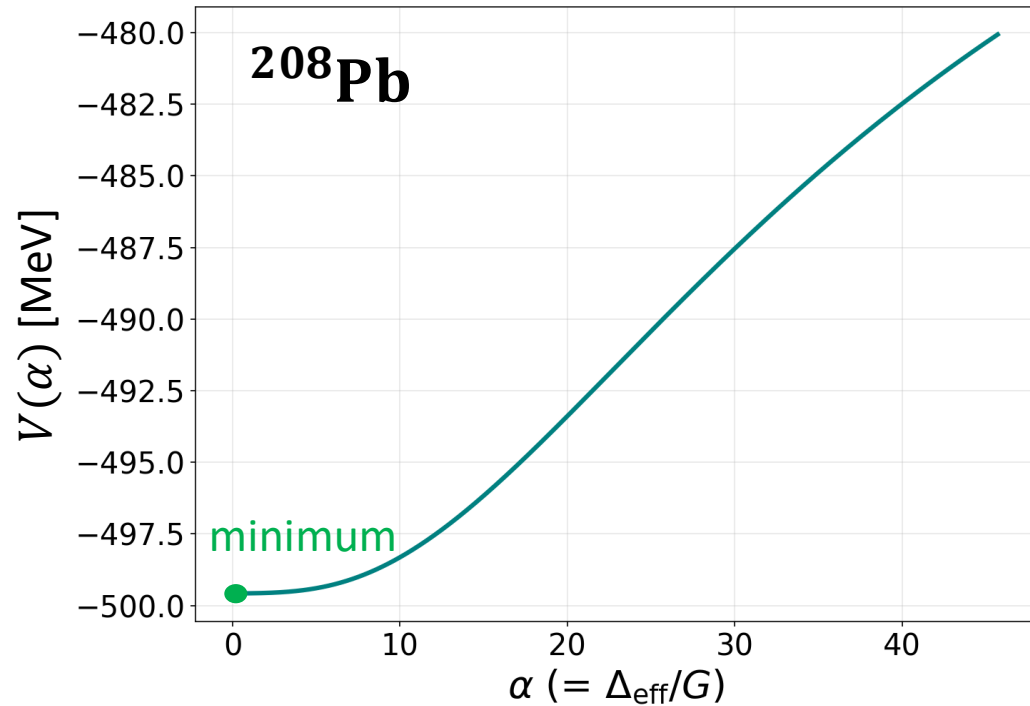
NG mode separation

Effect of residual interaction

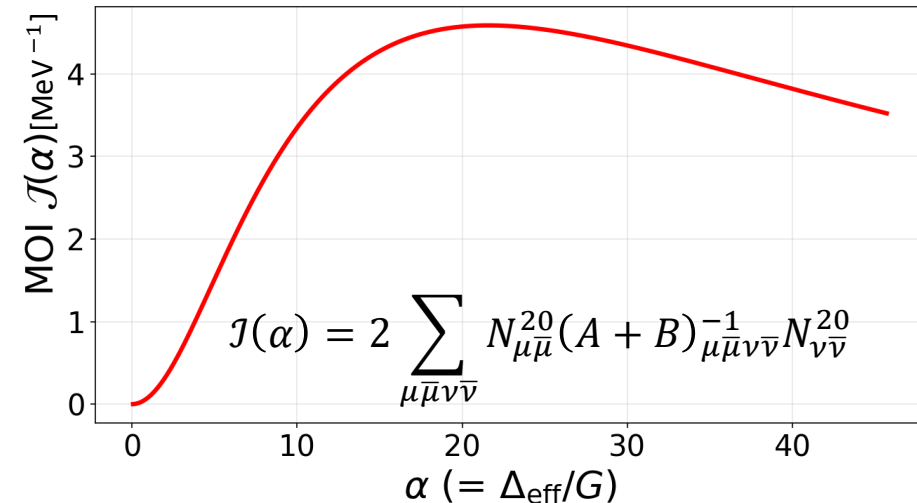
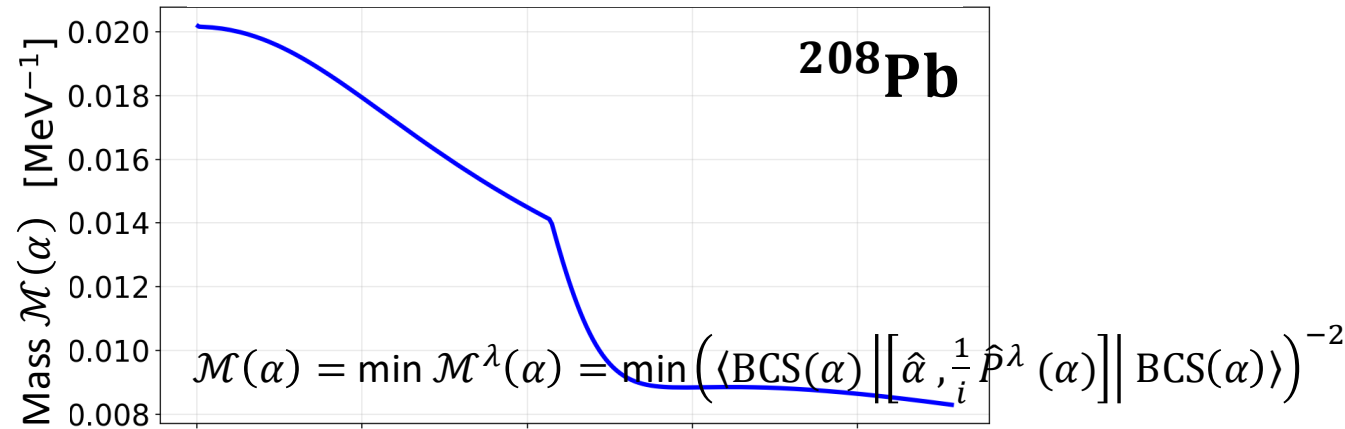
$$R_{n/2}^{\text{pair}}$$

Summary

Collective potential $V(\alpha)$, Mass $\mathcal{M}(\alpha)$, P-MOI $\mathcal{J}(\alpha)$



$$V(\alpha) = \langle \text{BCS}(\alpha) | \hat{H} | \text{BCS}(\alpha) \rangle$$



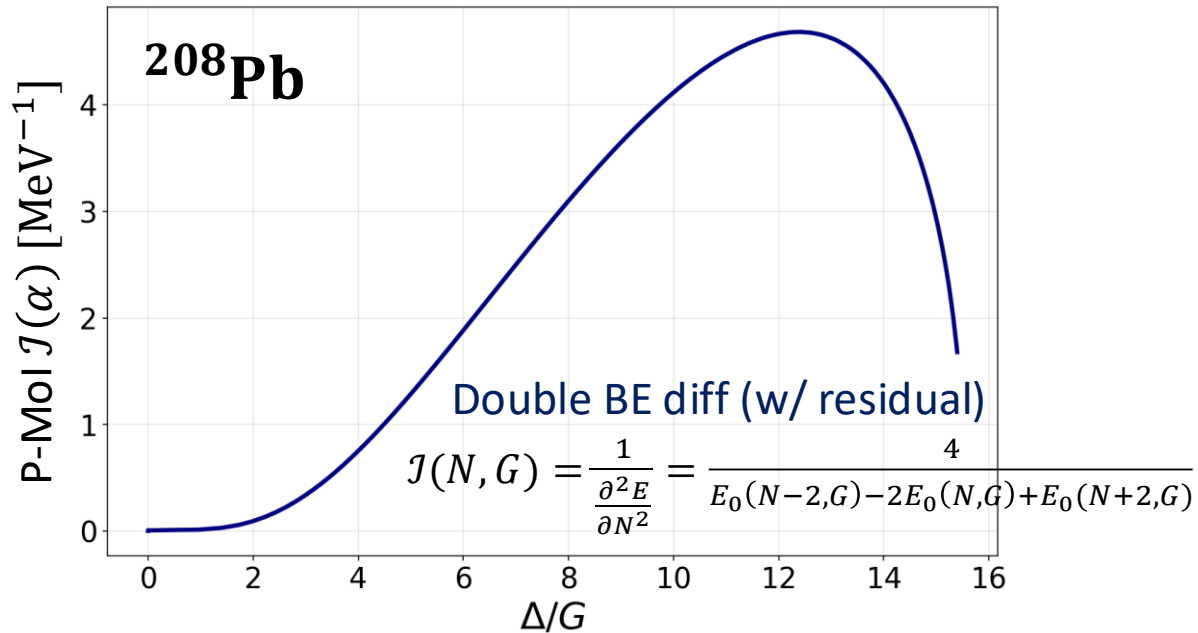
- The collective potential exhibits characteristics of the normal phase, but it is close to the critical phase.
- The pairing vibrational mode changes at $\alpha \sim 22$ ($\Delta = 3.2$ MeV), where the vibrational mass has a kink.
- P-MOI increases with α , reaches a maximum, and then decreases.

P-MOI $\mathcal{J}(\alpha)$

- P-MOI increases with α , reaches a maximum, and then decreases in the (subshell) closed nuclei.
→ This behavior is consistent with our previous study within the non-constrained BCS calculation.

(Ruike, Hinohara, and Nakatsukasa, arXiv:2504.11908)

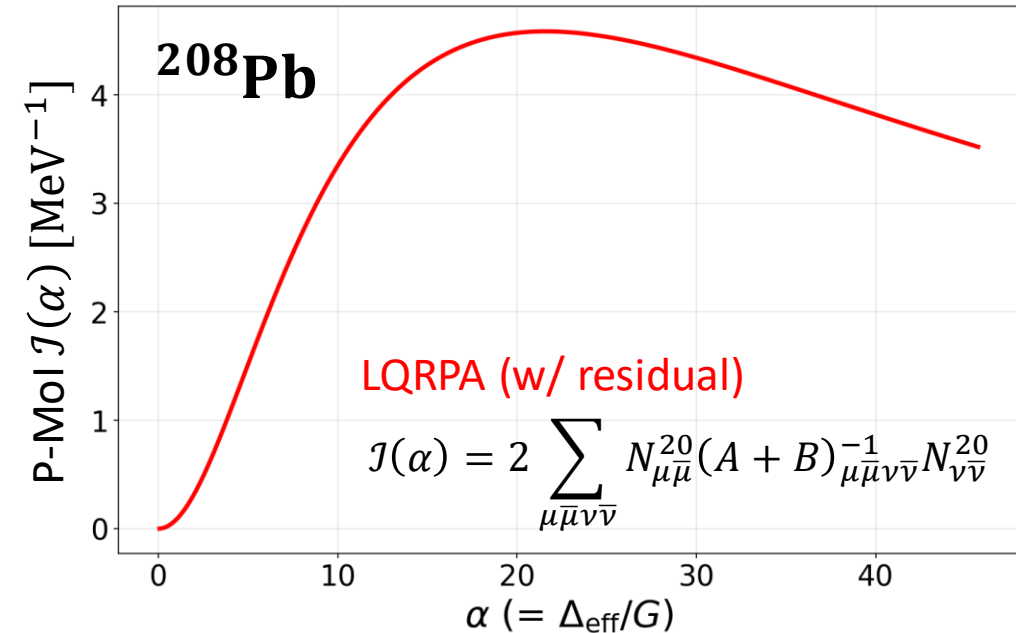
Our previous work (non-constrained BCS + double BE diff)



We changed the pairing strength G and solved the non-constrained Hamiltonian with BCS w.f.

$$H' = \sum_{\mu} e_{\mu} (c_{\mu}^{\dagger} c_{\mu} + c_{\bar{\mu}}^{\dagger} c_{\bar{\mu}}) - \frac{G}{2} \sum_{\mu\nu} (c_{\mu}^{\dagger} c_{\bar{\mu}}^{\dagger} c_{\bar{\nu}} c_{\nu} + c_{\bar{\mu}} c_{\mu} c_{\nu}^{\dagger} c_{\bar{\nu}}^{\dagger}) - \lambda (\hat{N} - N_0)$$

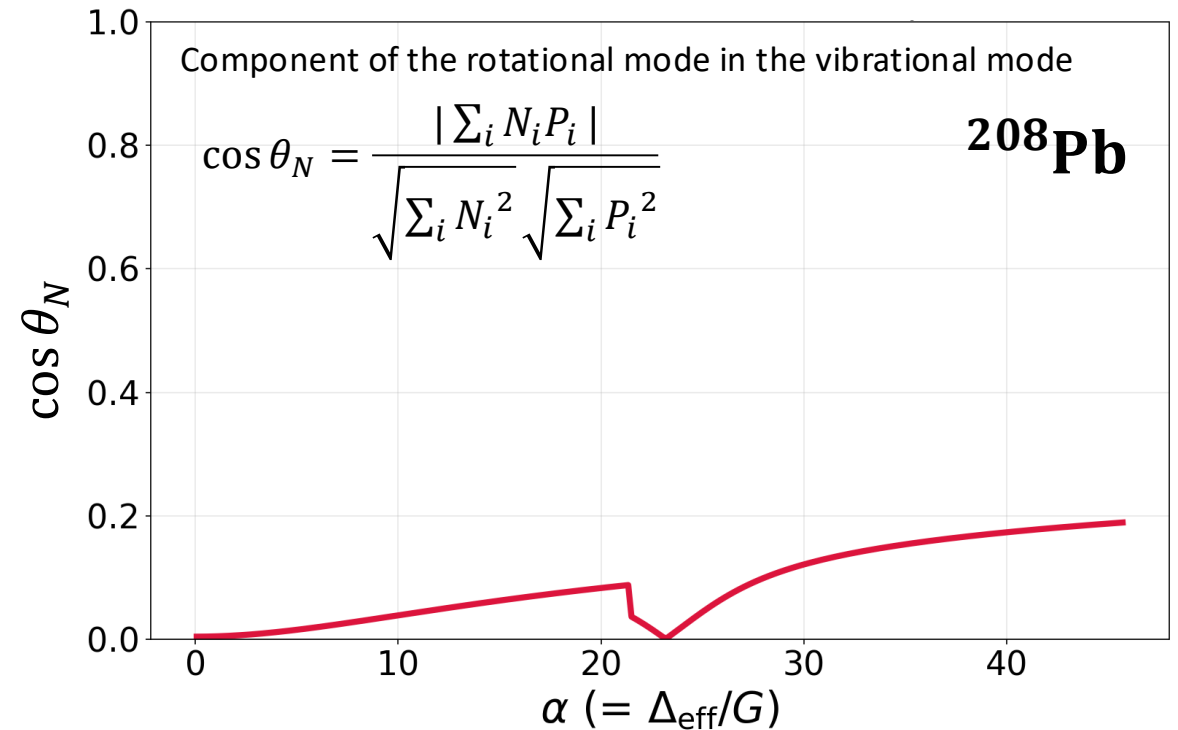
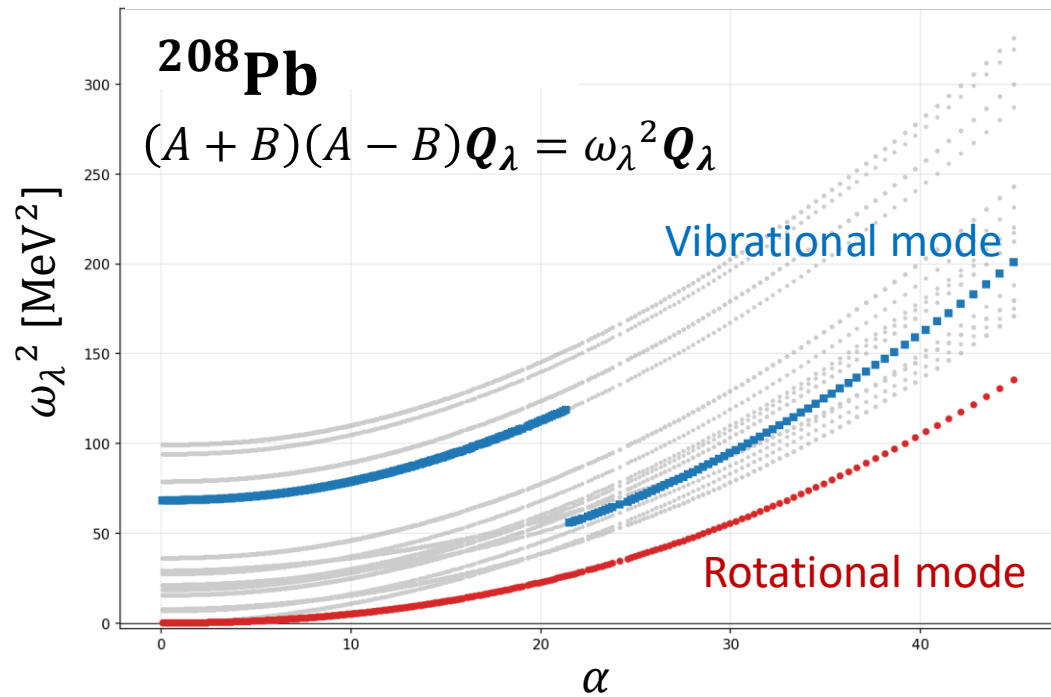
This work (α -constrained BCS + LQRPA)



We changed the order parameter α and solved the α -constrained Hamiltonian with BCS w.f.

$$H'' = H' + \mu(\hat{\alpha} - \alpha_0) + C(\hat{\alpha} - \alpha_0)^2$$

Mixing between rotational mode and vibrational modes



- In almost the entire region (except near $\alpha = 0$), the component of the rotational mode in the vibrational mode, $\cos \theta_N$, is nonzero.
→ We cannot separate the rotational and vibrational modes at QRPA level because the pair operator $\hat{\alpha}$ does not commute with the number operator $[\hat{\alpha}, \hat{N}] \neq 0$. (Hinohara et al. (2007))
- For axial **spatial rotation**, **$SO(3)$** is broken but **$SO(2)$** survives, so K remains a good quantum number and the vibrational and NG (rotational) modes are separated by K . For the **superfluid case**, **$U(1)$** is broken with **no other remaining symmetry**, so no quantum number is conserved, and the vibrational and NG (pairing rotational) modes can mix.

NG-mode separation

We orthogonalized P and Q vectors against the NG mode to obtain $\tilde{P}$ and $\tilde{Q}$.

T. Nakatsukasa, T. Inakura, and K. Yabana, Physical Review C 76, 024318 (2007).

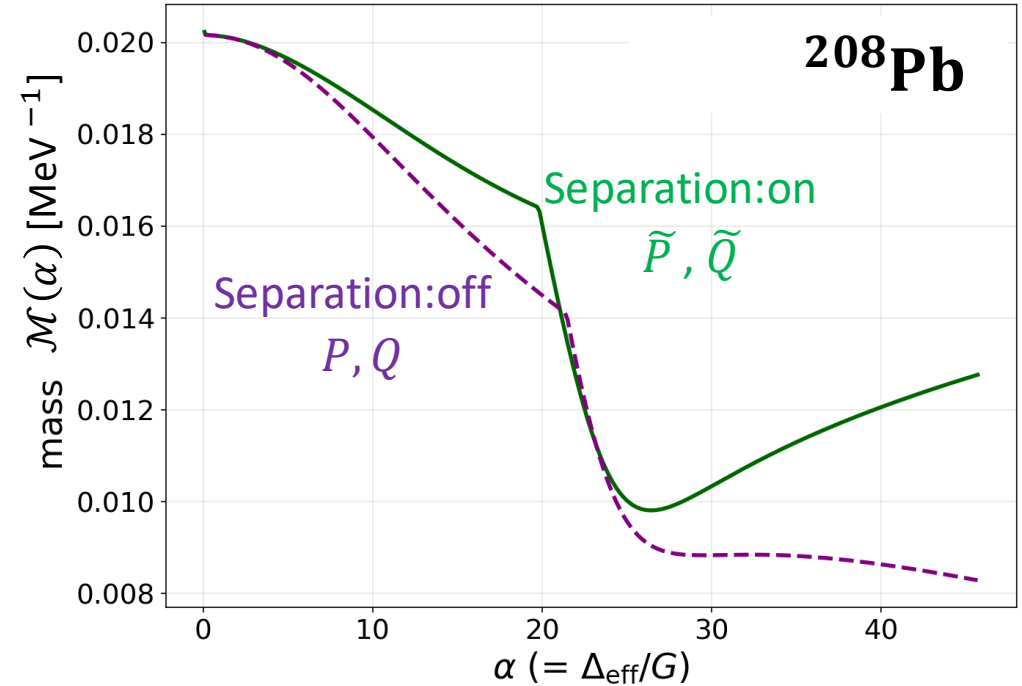
$$\begin{cases} \tilde{P}^\lambda = P^\lambda - b^\lambda \phi \\ \tilde{Q}^\lambda = Q^\lambda - a^\lambda N \end{cases} \quad \lambda: \text{index of QRPA mode}$$

QRPA commutation relations

$$[\hat{\tilde{Q}}, \hat{\phi}] = [\hat{\tilde{P}}, \hat{N}] = 0, \quad [\hat{\phi}, \hat{N}] = i$$

$$\begin{cases} a^\lambda = \sum_i Q_i^\lambda \phi_i \\ b^\lambda = \sum_i P_i^\lambda N_i \end{cases} \quad i: \text{vector index of } P^\lambda$$

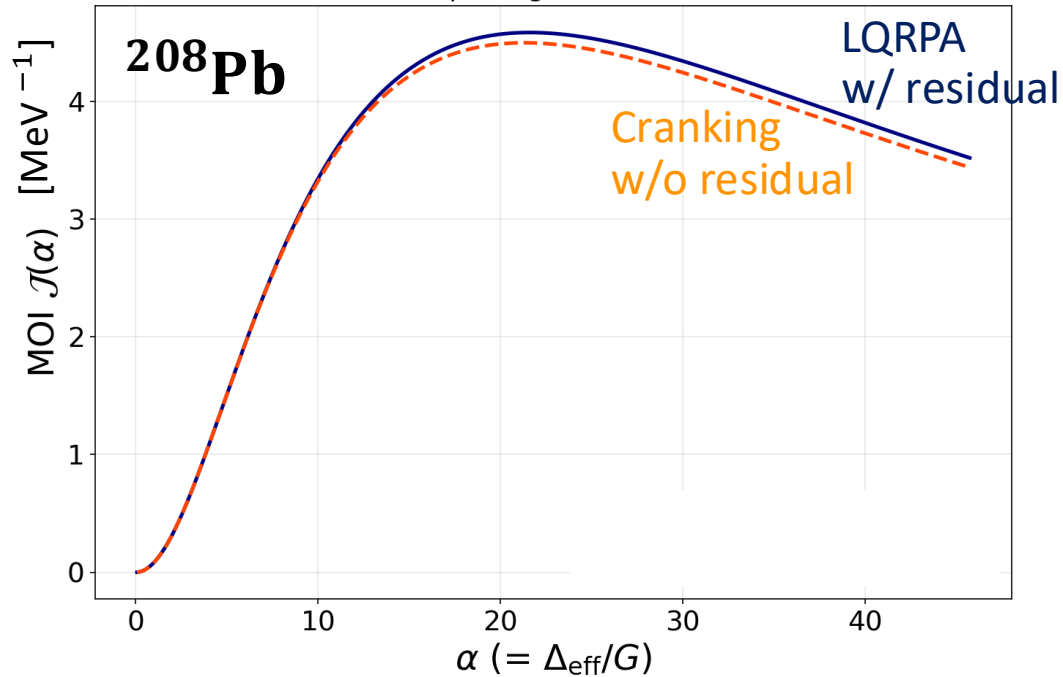
This prescription does not change Thouless-Valatin P-Mol.



$$\mathcal{M}(\alpha) = \min \mathcal{M}^\lambda(\alpha) = \min \left(\langle \text{BCS}(\alpha) | \left[\hat{\alpha}, \frac{1}{i} \hat{P}^\lambda(\alpha) \right] | \text{BCS}(\alpha) \rangle \right)^{-2}$$

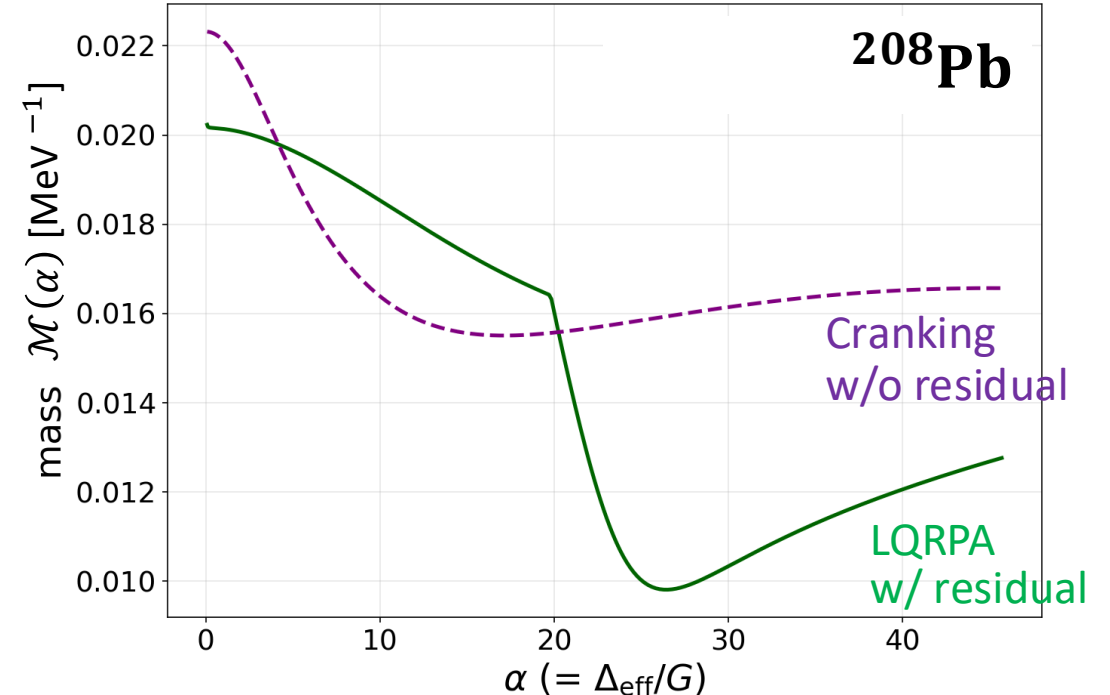
- We succeed in separating NG mode from all QRPA mode (NG mode mixing rate in the vibrational modes become zero).
- NG-mode separation prescription generally amplifies the pair vibrational mass.

Effect of residual interaction



LQRPA: $\mathcal{J}(\alpha) = 2 \sum_{\mu\bar{\mu}\nu\bar{\nu}} N_{\mu\bar{\mu}}^{20} (A+B)_{\mu\bar{\mu}\nu\bar{\nu}}^{-1} N_{\nu\bar{\nu}}^{20}$

Cranking: $\mathcal{J}(\alpha) = 4 \sum_{\mu} \frac{U_{\mu} V_{\mu}}{E_{\mu}}$

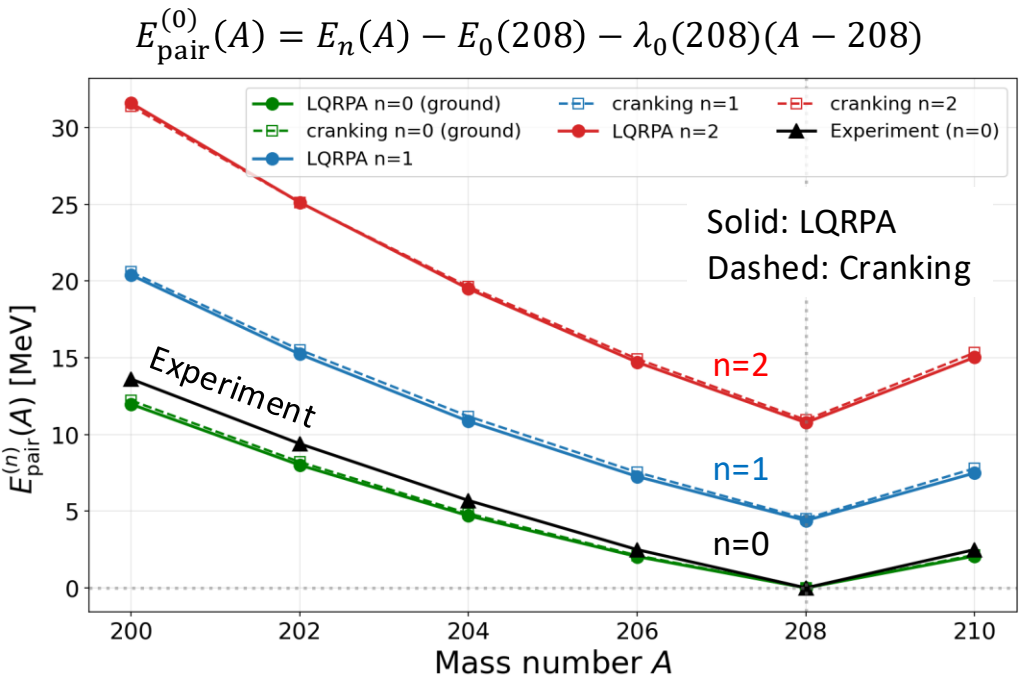


LQRPA: $\mathcal{M}(\alpha) = \min \mathcal{M}^{\lambda}(\alpha) = \min \left(\langle \text{BCS}(\alpha) | \left[\hat{\alpha}, \frac{1}{i} \hat{P}^{\lambda}(\alpha) \right] | \text{BCS}(\alpha) \rangle \right)^{-2}$

Cranking: $\mathcal{M}(\alpha) = \frac{\sum_{\mu} (U_{\mu}^2 - V_{\mu}^2)^2 / E_{\mu}^3}{2 [\sum_{\mu} (U_{\mu}^2 - V_{\mu}^2)^2 / E_{\mu}]^2}$

- The residual interaction affects the behavior of the pair vibrational mass, but it does not affect the behavior of P-Mol.

Effect of residual interaction



Normalized pair transfer intensity σ

	Expt.	LQRPA	Cranking	Difference: LQRPA-Cranking
$^{210}\text{Pb} \rightarrow ^{208}\text{Pb}$		0.9782	0.9762	+0.0020
$^{208}\text{Pb} \rightarrow ^{206}\text{Pb}$	1.00(10)	1.0000	1.0000	+0.0000
$^{206}\text{Pb} \rightarrow ^{204}\text{Pb}$	1.71(17)	1.9393	1.9865	-0.0472
$^{204}\text{Pb} \rightarrow ^{202}\text{Pb}$	2.20(22)	2.8280	2.9558	-0.1278
$^{202}\text{Pb} \rightarrow ^{200}\text{Pb}$		3.6297	3.8673	-0.2376

Clark and Macchiavelli, PRC **77**, 057301 (2008)

$$\sigma_{N+2 \rightarrow N} = \frac{\langle \psi_N | \hat{\alpha} | \psi_{N+2} \rangle^2}{\langle \psi_{206\text{Pb}} | \hat{\alpha} | \psi_{208\text{Pb}} \rangle^2}, \quad \hat{\alpha} = \frac{1}{2} \sum_{\mu} (c_{\mu}^{\dagger} c_{\bar{\mu}}^{\dagger} + c_{\bar{\mu}} c_{\mu})$$

- We cannot see the effect of residual interaction in the pair excitation energy.
- The residual interaction suppresses the value of the normalized pair transfer intensity.

We compare $R_{\Delta N/2}^{\text{pair}}$ that is calculated with PCH and with the Skyrme HFB.

$$R_{\Delta N/2}^{\text{pair}} = E_{\text{pair}}^{(0)}(N) / E_{\text{pair}}^{(0)}(2)$$

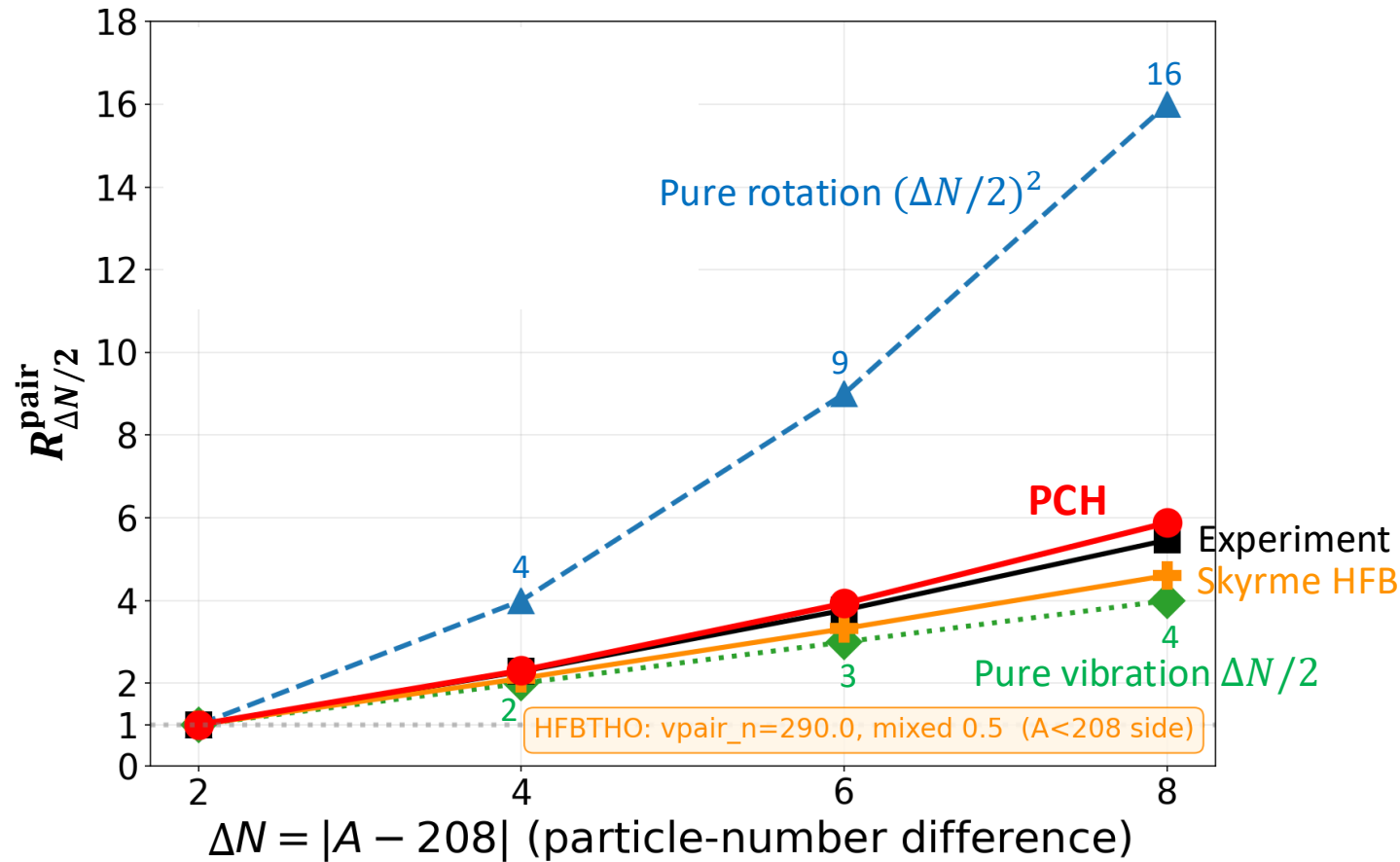
$$E_{\text{pair}}^{(0)}(A) = E_0(A) - E_0(208) - \lambda_0(208)(A - 208)$$

PCH: E_0 and λ_0 are calculated with the g.s. of PCH

Skyrme HFB: E_0 and λ_0 are calculated with HFB state

Skyrme HFB

- SLy4
- Pairing type: mixed, $V_{\text{pair_n}} = 290.0 \text{ MeV} \cdot \text{fm}^3$ (fits Δ in ^{188}Pb)



- Since PCH explains the experimental data better than the Skyrme HFB, we should take the pairing collective motion into account.

Outline

Introduction

Method

Results

Summary

Summary and Future Work

Summary and future work

Summary

- We have constructed the **pairing collective Hamiltonian** microscopically using the constrained BCS + local QRPA method.
- We have reproduced the experimental data of the pair transfer intensity.
- We cannot separate the rotational and vibrational mode at the QRPA level because the pair operator $\hat{\alpha}$ does not commute with the number operator $[\hat{\alpha}, \hat{N}] \neq 0$.
→ We adopt the **NG-mode separation prescription**.
- The **residual interaction suppresses** the value of the normalized **pair transfer intensity**.
- From the $R_{n/2}^{\text{pair}}$ perspective, we should take pairing collective motion into account.

Future work

- We are now constructing the PCH based on the **Skyrme DFT + FAM-QRPA** calculation to analyze the magicity in a wide region of the nuclear chart.

Back up

魔法核近辺におけるペアリング 大振幅集団運動の微視的記述

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Q&A

- (1) order parameters: $\langle P \rangle$ 、複素数になるように取るべき（今は $\langle P+P^\dagger \rangle$ をorderparameterと呼んで実数に限定してしまっている
- (2) collective variables: $|\langle P \rangle|$ と $\langle P \rangle$ の位相の2種類。波動関数を分離型に置くとCHには陽に現れない。 $|\langle P \rangle|$ の求め方はBESのやり方でも $\langle P+P^\dagger \rangle$ として実数部分だけ取ってきててもどっちでもいい
- (3) constraint operator: 今はエルミートなpp(hh)typeの1体演算子にしたかったので $\langle P+P^\dagger \rangle$ としたがこれだとNと交換せずQRPAの理念を守っていない。正確にはASCCをして正しいconstraint operatorを求めるべきだがコストが高いため、近似的にph型の（Nと可換な）演算子で表現できないか模索中。

結局collective potentialの形はcriticalなのか？

→現状はペアリング相互作用だけ入れて模型空間も限定しているので断定はできない。

今後Skyrmeなどで一般的な有効相互作用を入れても実験値と合うことを確かめた上で言及する

しかしmassとmoiの情報から真のcollective coordinateに対する $V(q)$ が計算できて、定性的にはよりcriticalになることが予想されている

Pair transfer intensity

Clark and Macchiavelli, PRC **77**, 057301 (2008)

R. A. Broglia, R. J. Liotta, B. S. Nilsson, and E. R. Flynn, Nucl. Phys. A343, 24 (1980)

測定反応: $^{208}\text{Pb}(p,t)^{206}\text{Pb}$, $^{206}\text{Pb}(p,t)^{204}\text{Pb}$, $^{204}\text{Pb}(p,t)^{202}\text{Pb}$ という** (p,t) 反応(2中性子ピックアップ反応)**の微分断面積測定

測定角度: $\theta=30^\circ$ (differential cross sectionのピーク角度)

W. A. Lanford and J. McGrory, Phys. Lett. 54B (1973) 238 から得たデータ

微分断面積 $d\sigma/d\Omega$ (mb/sr単位)

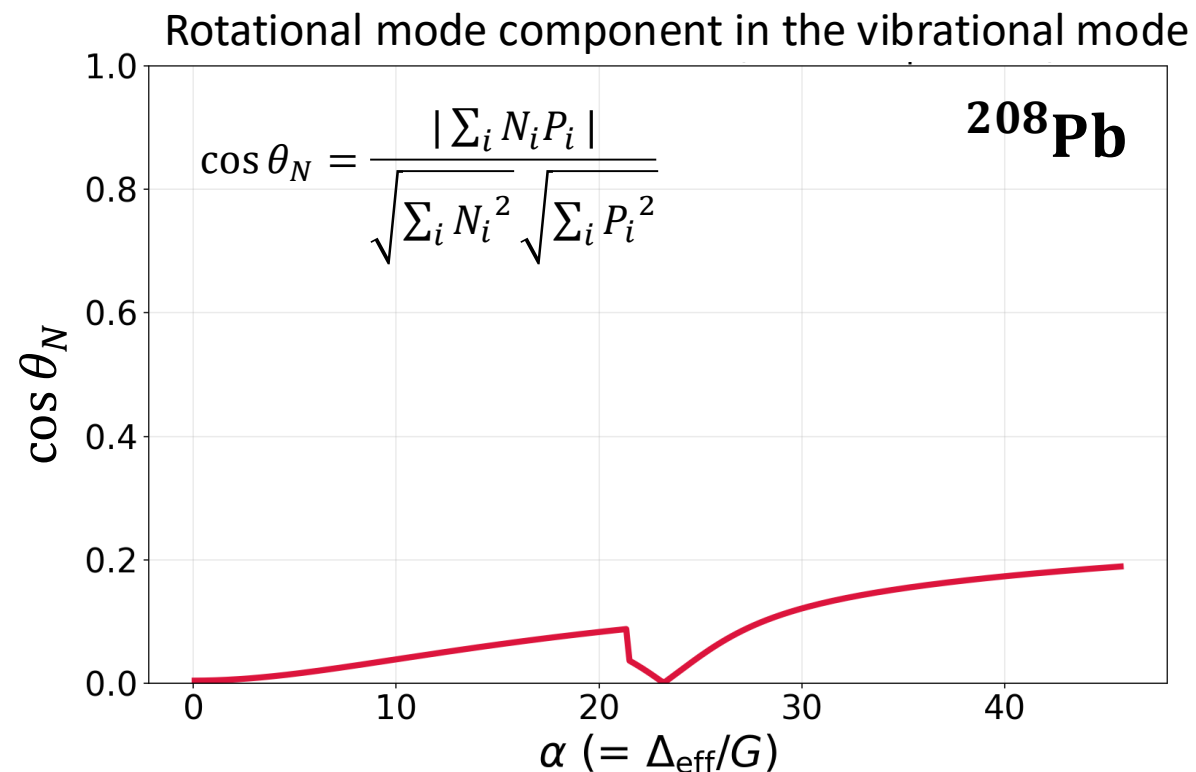
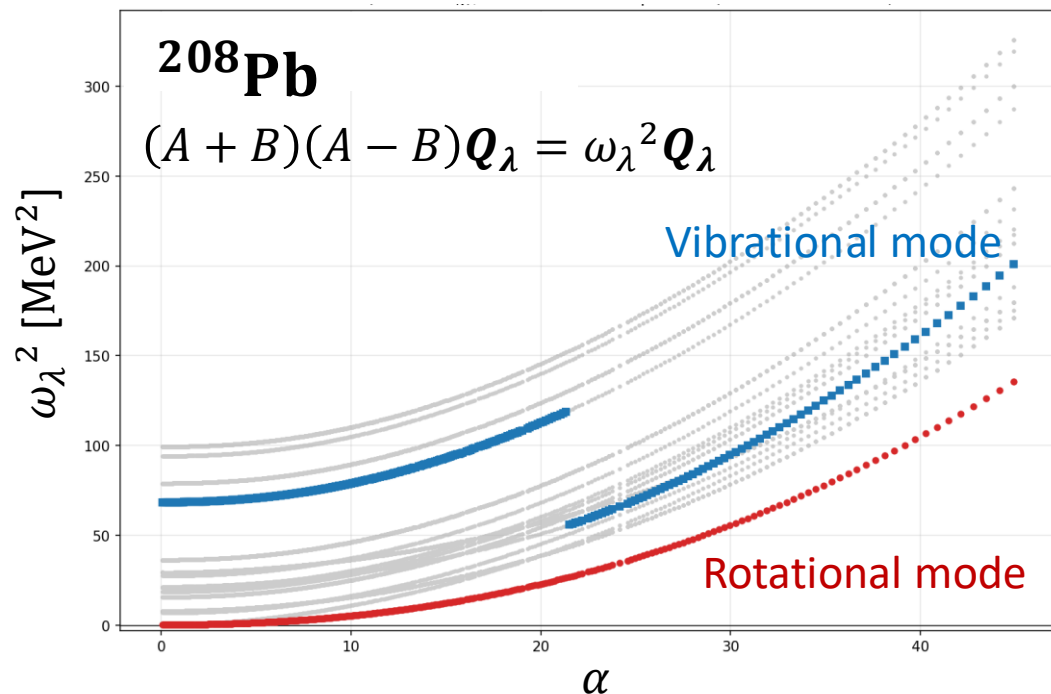
208→206: $0.082 \pm 15\%$

206→204: $0.14 \pm 15\%$

204→202: $0.18 \pm 15\%$

Normalized pair transfer intensity = $(d\sigma(\theta; A(p,t)(A-2))/d\Omega) / (d\sigma(\theta; 208\rightarrow 206)/d\Omega)$

Mixing between rotational mode and vibrational modes



- In the case of the **spatial rotation** associated with axial deformation, although the **SO(3)** symmetry is broken, the **SO(2)** symmetry remains. Consequently, the K quantum number is conserved, and the surface vibrational mode and the NG (rotational) mode belong to different K values.
- In the **superfluid phase**, **U(1)** symmetry is broken and there is **no other remaining symmetry**, so no quantum number is conserved, and the surface vibrational mode and the NG (pairing rotational) mode can mix.

Types of pairing interaction

■ 4 types of pairing interaction

Under exchange of the two particles

Spin ^{Orbit} Total	Orbit L	Spin S	Isospin T
1S_0 Antisymmetric	Symmetric Ex.) S-wave	Antisymmetric ($S=0$) (singlet:1)	Symmetric ($T=1$, isovector) ($p\downarrow p\downarrow$ or $n\uparrow n\uparrow$ or $p\downarrow n\uparrow$)
3P_2 Antisymmetric	Antisymmetric Ex.) P-wave	Symmetric ($S=1$) (triplet:3)	Symmetric ($p\downarrow p\downarrow$ or $n\uparrow n\uparrow$ or $p\downarrow n\uparrow$)
3S_1 Antisymmetric	Symmetric Ex.) S-wave	Symmetric (triplet:3)	Antisymmetric ($T=0$, isoscalar) ($p\downarrow n\uparrow$)
1P_1 Antisymmetric	Antisymmetric Ex.) P-wave	Antisymmetric (singlet:1)	Antisymmetric ($p\downarrow n\uparrow$)

Dominant in the ground state

In neutron stars

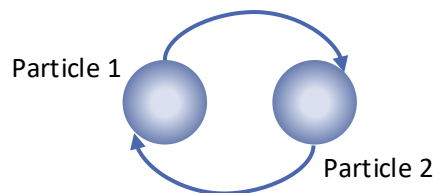
■ Total orbital angular momentum

Symmetric: The relative orbital wave function of the two nucleons is symmetric under their exchange.

→Parity of the relative orbital angular momentum is **even** (s,d,g,... wave)

Antisymmetric: The relative orbital wave function of the two nucleons is antisymmetric under their exchange.

→Parity of the relative orbital angular momentum is **odd** (p,f,h,... wave)



Theoretical definition in this study

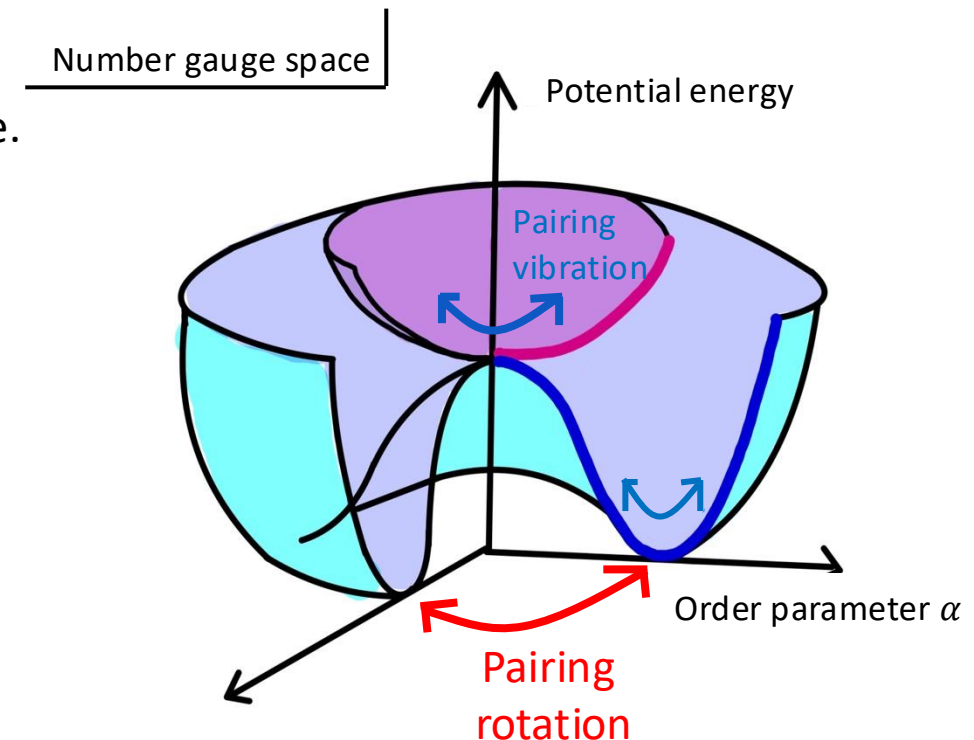
Pairing vibrational mode : The collective mode that has the largest overlap with the **pair collective coordinate α**

→ By analogy with surface vibrations in ordinary space, this mode is defined only near the potential minimum.

Pairing rotational mode : The collective mode that has the largest overlap with the **particle number N**

→ Pairing rotational mode as the NG mode is defined only at the potential minimum of the superfluid phase

→ As in the case of spatial rotation, the pairing rotational energy appears in the laboratory frame.



- To analyze various situations including the critical point, it is necessary to construct the pairing collective Hamiltonian that can describe the pairing rotation and vibration together

$$\hat{H} = \hat{V}(\alpha) + \hat{T}_{\text{rot}}(\alpha) + \hat{T}_{\text{vib}}(\alpha) = V(\alpha) + \frac{\hat{N}^2}{2\mathcal{I}(\alpha)} - \frac{\partial}{\partial \alpha} \frac{1}{2\mathcal{M}(\alpha)} \frac{\partial}{\partial \alpha}$$

α : order parameter of the pairing phase transition

- Present approaches for pairing collective Hamiltonians rely on simple monopole pairing interactions, and therefore can be applied only to a limited region in the nuclear chart.

(D. R. Bés, R. A. Broglia, R. P. J. Perazzo, and K. Kumar, Nucl. Phys. A, 143, 1-33 (1970))

- Bés et al. (1970) used cranking masses **without the residual interaction**, while Clark et al. (2008) used **constant masses**.

Pairing collective Hamiltonian in D. R. Bés et al. (1970)

$$\hat{H}_{\text{coll}} = -\frac{1}{2\mathcal{M}(\alpha)} \frac{\partial^2}{\partial \alpha^2} - \frac{1}{4} \left(\frac{1}{\mathcal{I}(\alpha)\mathcal{M}(\alpha)} \frac{\partial \mathcal{I}(\alpha)}{\partial \alpha} - \frac{1}{\mathcal{M}(\alpha)^2} \frac{\partial \mathcal{M}(\alpha)}{\partial \alpha} \right) \frac{\partial}{\partial \alpha} + \left(V(\alpha) + \frac{(N-N_0)^2}{2\mathcal{I}(\alpha)} \right)$$

→we use dynamical collective masses obtained from QRPA calculations, which **include the residual interaction**

We performed the constrained BCS calculations to obtain the state including the **neutron 1S_0 pairing** in **Pb** isotopes.

Original Hamiltonian

$$H = \sum_{\mu} e_{\mu} (c_{\mu}^{\dagger} c_{\mu} + c_{\bar{\mu}}^{\dagger} c_{\bar{\mu}}) - \frac{G}{2} \sum_{\mu\nu} (c_{\mu}^{\dagger} c_{\bar{\mu}}^{\dagger} c_{\bar{\nu}} c_{\nu} + c_{\bar{\mu}} c_{\mu} c_{\bar{\nu}}^{\dagger} c_{\nu}^{\dagger}).$$

Pairing term $\bar{\mu}$: time-reversed orbit of μ

We constrain the Hamiltonian with the particle number N and the pairing order parameter α .

$\bar{\mu}$: time-reversed orbit of μ

Linear constraint

Augmented Lagrangian Method
(A. Staszczak et al. (2010))

Pair operator ($\alpha = \langle \hat{\alpha} \rangle$):

$$\hat{\alpha} = \frac{1}{2} \sum_{\mu\nu} (c_{\mu}^{\dagger} c_{\bar{\mu}}^{\dagger} + c_{\bar{\nu}} c_{\nu})$$

We solve the variational equation using the BCS wavefunctions.

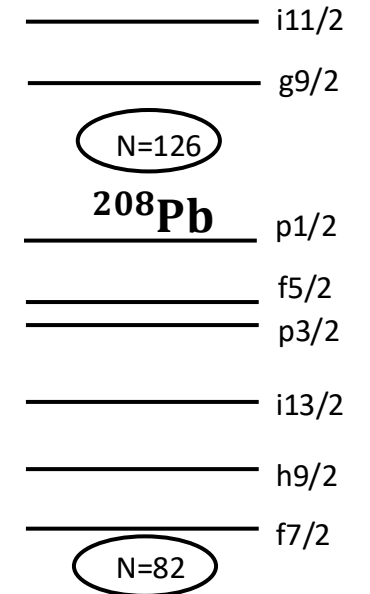
(We solve the BCS gap equation for Δ including the α -constraint terms.)

$$\Delta_{\text{eff}} = G \langle \alpha \rangle - \frac{\mu}{2} - \mathbf{C}(\langle \alpha \rangle - \alpha_0)$$

Here, we determine the pairing strength **G** so as to reproduce the experimental **pairing gap Δ of ^{188}Pb** .

Then we obtain the potential $V(\alpha)$. $V(\alpha) = \langle \text{BCS}(\alpha) | \hat{H} | \text{BCS}(\alpha) \rangle$

Model space (Pb)



QRPA : describes harmonic collective excitations around the potential minimum in terms of the collective coordinate

→ This is valid when the potential is quadratic (small-amplitude dynamics)

Local QRPA : calculate the collective excitation in each collective coordinate

→ We can include the effects of the potential beyond quadratic order
(→ large-amplitude dynamics)

Since the pairing potential is soft, we use the local QRPA.

Phonon operator

$$\Gamma_{\lambda}^{\dagger} = \sum_{ij} (X_{ij}^{\lambda} \beta_i^{\dagger} \beta_j^{\dagger} - Y_{ij}^{\lambda} \beta_i \beta_j)$$

quasiparticle operator

We extract the residual part of the α constraint terms because it was neglected at the BCS level.

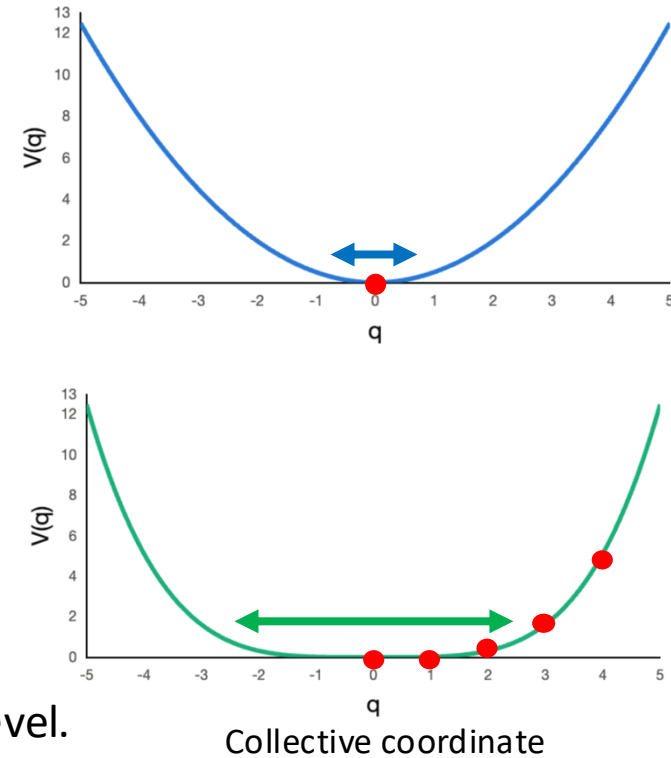
$$[H' + (\mu(\hat{\alpha} - \alpha_0) + \mathbf{C}(\hat{\alpha} - \alpha_0)^2)_{\text{MF}}, \Gamma_{\lambda}^{\dagger}] = \Gamma_{\lambda}^{\dagger}$$

From this equation, we obtain the A and B matrices, and then diagonalize the QRPA equation.

$$\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \omega_{\lambda} \begin{pmatrix} X \\ -Y \end{pmatrix}$$

$$(A + B)(A - B)\mathbf{Q}_{\lambda} = \omega_{\lambda}^2 \mathbf{Q}_{\lambda} \qquad -i(A + B)\mathbf{P}_{\lambda} = \omega_{\lambda}^2 \mathbf{Q}_{\lambda}$$

We normalize the X , Y and P , Q vectors.



Local QRPA : calculate the collective excitation in each collective coordinate using the constrained mean-field (BCS) state.
(→large-amplitude dynamics)

Since the pairing potential is soft, we use the local QRPA.

QRPA equation in each collective coordinate

$$(A + B)(A - B)Q_\lambda = \omega_\lambda^2 Q_\lambda \quad -i(A + B)P_\lambda = \omega_\lambda^2 Q_\lambda$$

Thouless-Valatin pairing moment of inertia (**P-MOI**, $\mathcal{J}(\alpha)$)

$$\mathcal{J}_{TV}(\alpha) = 2 \sum_{\mu, \nu > 0} N_{\mu\bar{\mu}}^{20}(\alpha) (A(\alpha) + B(\alpha))^{-1}_{\mu\bar{\mu}\nu\bar{\nu}} N_{\nu\bar{\nu}}^{20}(\alpha)$$

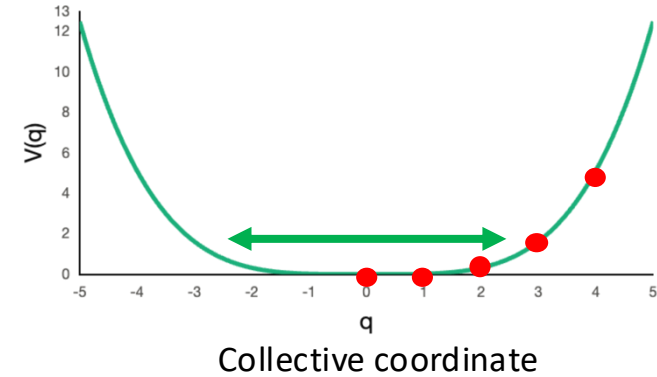
Pairing vibration mass ($\mathcal{M}(\alpha)$)

$$\mathcal{M}(\alpha) = \min \mathcal{M}^\lambda(\alpha) = \min \left(\langle \text{BCS}(\alpha) \left| \left[\hat{a}, \frac{1}{i} \hat{P}^\lambda(\alpha) \right] \right| \text{BCS}(\alpha) \rangle \right)^{-2}$$

Washiyama, Hinohara, and Nakatsukasa, PRC **103**, 014306 (2021)

We insert $V(\alpha)$, $\mathcal{J}(\alpha)$, and $\mathcal{M}(\alpha)$ of ^{208}Pb into the PCH.

$$\hat{H}_{\text{coll}} = -\frac{1}{2\mathcal{M}(\alpha)} \frac{\partial^2}{\partial \alpha^2} - \frac{1}{4} \left(\frac{1}{\mathcal{J}(\alpha)\mathcal{M}(\alpha)} \frac{\partial \mathcal{J}(\alpha)}{\partial \alpha} - \frac{1}{\mathcal{M}(\alpha)^2} \frac{\partial \mathcal{M}(\alpha)}{\partial \alpha} \right) \frac{\partial}{\partial \alpha} + (V(\alpha) + \lambda(N - N_0) + \frac{(N - N_0)^2}{2\mathcal{J}(\alpha)})$$



Thouless-Valatin moment of inertia (pairing version)

$$\mathcal{J}_{\text{TV}}(\alpha) = 2 \sum_{\mu, \nu > 0} N_{\mu\bar{\mu}}^{20}(\alpha) (A(\alpha) + B(\alpha))_{\mu\bar{\mu}\nu\bar{\nu}}^{-1} N_{\nu\bar{\nu}}^{20}(\alpha)$$

$$\mathcal{J}_{\text{TV}}(\alpha) = 2 \sum_k \frac{\overbrace{|\langle \Psi_k(\alpha) | N_k^{20}(\alpha) \rangle|^2}^{\text{Eigenvector of } (A+B) \text{ matrix}}}{\underbrace{\lambda_k(\alpha)}_{\text{Eigenvalue of } (A+B) \text{ matrix}}}$$

$$(A + B) |\Psi_k\rangle = \lambda_k |\Psi_k\rangle$$

$$(A + B) = \sum_k \lambda_k |\Psi_k\rangle \langle \Psi_k|$$

P-MOI: The sum of the overlap between all modes and N, weighted by their eigenvalues.

Local QRPA calculation (Pairing vibration mass $\mathcal{M}(\alpha)$) method 9/18

Collective mass : The coefficient of the kinetic term when the Hamiltonian is expressed in terms of the collective coordinate

$$T = \frac{1}{2} \sum_i \dot{q}_i^2 = \frac{1}{2} \sum_j \mathcal{M}_j(s) \dot{s}_j^2$$

Original coordinate of each nucleon Induced collective coordinate

We choose the pairing vibrational mode that has the strongest coupling to the pair field (most collective for α).

We take the smallest $\mathcal{M}^\lambda(\alpha)$ (largest overlap with $\hat{\alpha}$ and $\hat{P}$) among all $\mathcal{M}^\lambda(\alpha)$ calculated from the eigenvectors P^λ .

Hinohara et al. PRC **82**, 064313 (2010)

$$\mathcal{M}(\alpha) = \min \mathcal{M}^\lambda(\alpha) = \min \left(\langle \text{BCS}(\alpha) | \left[\hat{\alpha}, \frac{1}{i} \hat{P}^\lambda(\alpha) \right] | \text{BCS}(\alpha) \rangle \right)^{-2}$$

Washiyama, Hinohara, and Nakatsukasa, PRC **103**, 014306 (2021)

Local QRPA calculation (Pairing vibration mass $\mathcal{M}(\alpha)$) method 9/18

General way to change the kinetic energy with the collective coordinates q_i to with the other collective variable s .

$$T = \frac{1}{2} \sum_i \dot{q}_i^2 = \frac{1}{2} \sum_{mn} \mathcal{M}_{mn}(s) \dot{s}_m \dot{s}_n$$

The collective mass $\mathcal{M}_{mn}(s)$ is defined by this transformation of collective coordinates.

$$\mathcal{M}_{mn}(s) \equiv \sum_{ij} \frac{\partial q_i}{\partial s_m} \mathcal{M}_{ij}(s) \frac{\partial q_j}{\partial s_n} = \sum_i \frac{\partial q_i}{\partial s_m} \frac{\partial q_i}{\partial s_n}$$

These derivative is expressed by the $\hat{P}$ in the solution of local QRPA.

$$\frac{\partial s_m}{\partial q_i} = \frac{\partial}{\partial q_i} \langle \phi(s) | \widehat{s}_m | \phi(s) \rangle = \langle \phi(s) | \left[\widehat{s}_m, \frac{1}{i} \hat{P}_i(s) \right] | \phi(s) \rangle$$

$$\times \hat{P}(q) | \phi(q) \rangle = i \frac{\partial}{\partial q} | \phi(q) \rangle$$

Washiyama, Hinohara, and Nakatsukasa, PRC **103**, 014306 (2021)

In the case of the pairing collective mode, the collective variable s is α .

$$\alpha = \langle a^\dagger a^\dagger + a a \rangle / 2 = \text{Re}(\Delta) / G$$

$$\frac{\partial \alpha}{\partial q} = \langle \phi(\alpha) | \left[\hat{\alpha}, \frac{1}{i} \hat{P}(\alpha) \right] | \phi(\alpha) \rangle$$

$$\mathcal{M}_{mn}(\alpha) = \frac{\partial q}{\partial \alpha} \frac{\partial q}{\partial \alpha} = \left(\langle \text{CBCS}(\alpha) | \left[\hat{\alpha}, \frac{1}{i} \hat{P}(\alpha) \right] | \text{CBCS}(\alpha) \rangle \right)^{-2} = \frac{4}{\left(\sum_{\mu} \left(\frac{1}{i} P_{\mu\bar{\mu}}^{20} \alpha_{\mu\bar{\mu}}^{20*} - \frac{1}{i} P_{\mu\bar{\mu}}^{20*} \alpha_{\mu\bar{\mu}}^{20} \right) \right)^2}$$

Local QRPA calculation (Pairing vibration mass)

method 10/18

QRPA mode excitation operator

$$O_v^\dagger = \sum_{mn} \{X_v(mn)\beta_m^\dagger\beta_n^\dagger - Y_v(mn)\beta_n\beta_m\}$$

We solved the QRPA equation in the P,Q expression.

$$\widehat{P}_v = \frac{1}{i}\sqrt{M_v\Omega_v}\frac{1}{\sqrt{2}}(O_v - O_v^\dagger) \quad \widehat{Q}_v = \frac{1}{\sqrt{M_v\Omega_v}}\frac{1}{\sqrt{2}}(O_v^\dagger + O_v)$$

First, we solved this equation (diagonalized this matrix) and obtain the vector $\mathbf{Q}$.

$$(A + B)(A - B)\mathbf{Q} = \Omega_v^2\mathbf{Q}$$

Second, we obtain the vector $\mathbf{P}$ using this equation.

$$-i(A + B)\mathbf{P} = \Omega_v^2\mathbf{Q}$$

QRPA equation in P,Q expression

Finally we obtain the pairing vibration mass.

$$\mathcal{M}_v(\alpha) = \frac{4}{\left(\sum_{\mu} \left(\frac{1}{i}P_{\mu\bar{\mu}}^{20}\alpha_{\mu\bar{\mu}}^{20*} - \frac{1}{i}P_{\mu\bar{\mu}}^{20*}\alpha_{\mu\bar{\mu}}^{20}\right)\right)^2}$$

Referenced to Washiyama, Hinohara, and Nakatsukasa, PRC **103**, 014306 (2021)

✂ We take the smallest $\mathcal{M}_v(\alpha)$ in all $\mathcal{M}_v(\alpha)$ which is calculated with all eigen vectors P,Q.

Referenced to Hinohara et al. PRC **82**, 064313 (2010)

We insert $V(\alpha)$, $\mathcal{J}(\alpha)$, and $\mathcal{M}(\alpha)$ of ^{208}Pb into the PCH.

$$\hat{H}_{\text{coll}} = -\frac{1}{2\mathcal{M}(\alpha)} \frac{\partial^2}{\partial \alpha^2} - \frac{1}{4} \left(\frac{1}{\mathcal{J}(\alpha)\mathcal{M}(\alpha)} \frac{\partial \mathcal{J}(\alpha)}{\partial \alpha} - \frac{1}{\mathcal{M}(\alpha)^2} \frac{\partial \mathcal{M}(\alpha)}{\partial \alpha} \right) \frac{\partial}{\partial \alpha} + \frac{(V(\alpha) + \lambda(N - N_0) + \frac{(N - N_0)^2}{2\mathcal{J}(\alpha)})}{V_{\text{coll}}(\alpha, N)}$$

Then we diagonalize the PCH.

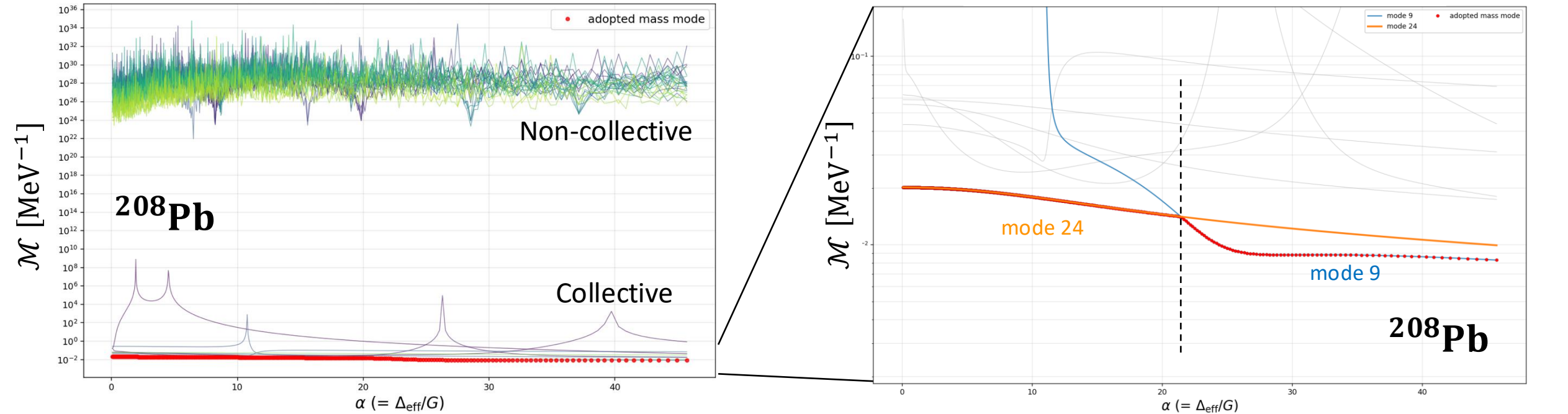
Boundary conditions

$$\begin{cases} \psi(\alpha_{\min}) = (-1)^N \psi(-\alpha_{\min}) & \text{for } ^{208}\text{Pb (the reference nucleus of the pairing collective motion)} \\ \psi(\alpha_{\min}) = 0 & \text{for } ^{200-206, 210}\text{Pb } (V_{\text{coll}} \rightarrow \infty, \alpha \rightarrow 0) \end{cases}$$

Normalization condition

$$\int_{\alpha_{\min}}^{\alpha_{\max}} \psi_N^*(\alpha) \psi_N(\alpha) \underbrace{\sqrt{\mathcal{J}(\alpha)\mathcal{M}(\alpha)} d\alpha}_{\text{Volume element}} = 1$$

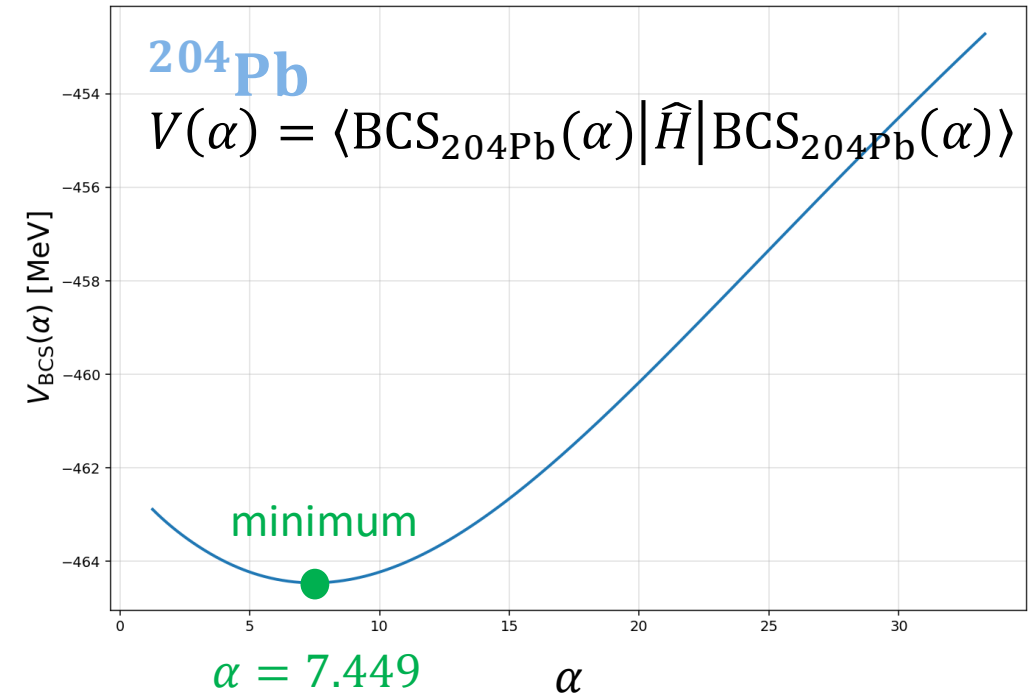
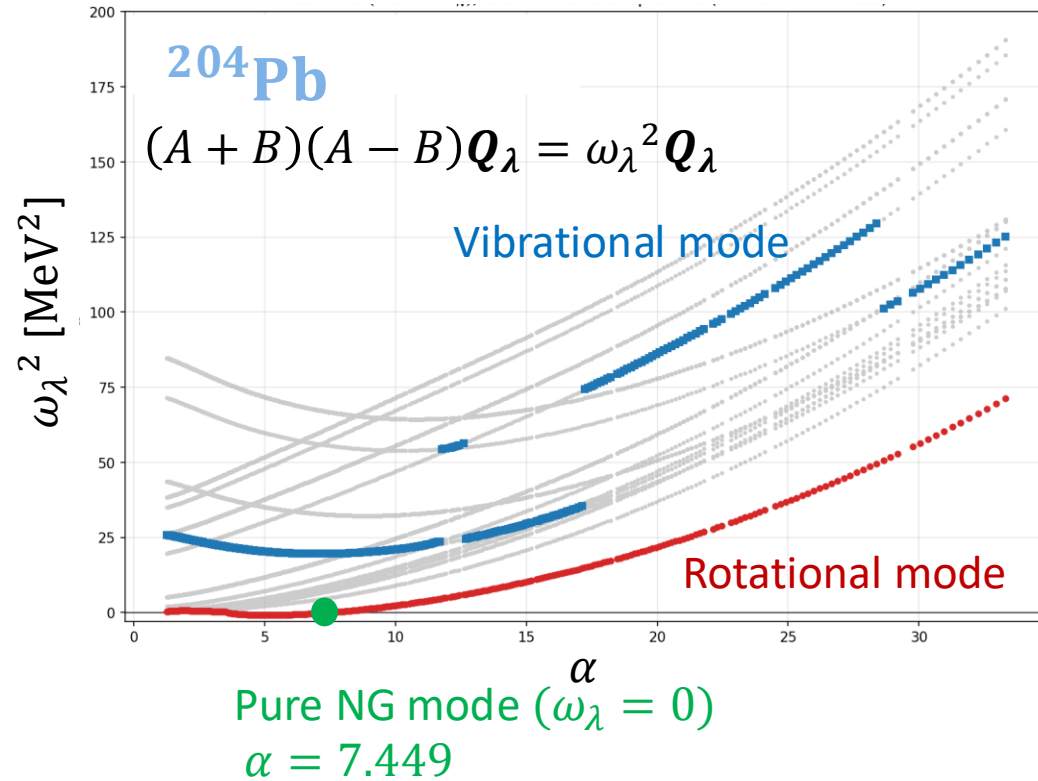
Pairing vibrational mode (smallest $\mathcal{M}(\alpha)$ mode)



$$\mathcal{M}(\alpha) = \min \mathcal{M}^\lambda(\alpha) = \min \left(\langle \text{BCS}(\alpha) | \left[\hat{a}, \frac{1}{i} \hat{P}^\lambda(\alpha) \right] | \text{BCS}(\alpha) \rangle \right)^{-2}$$

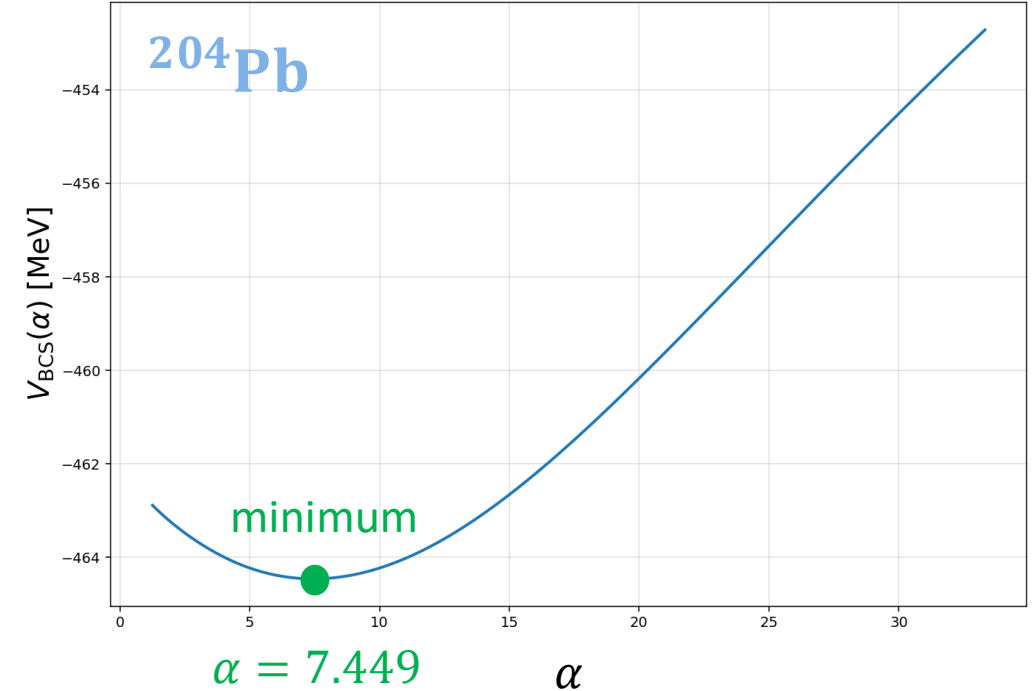
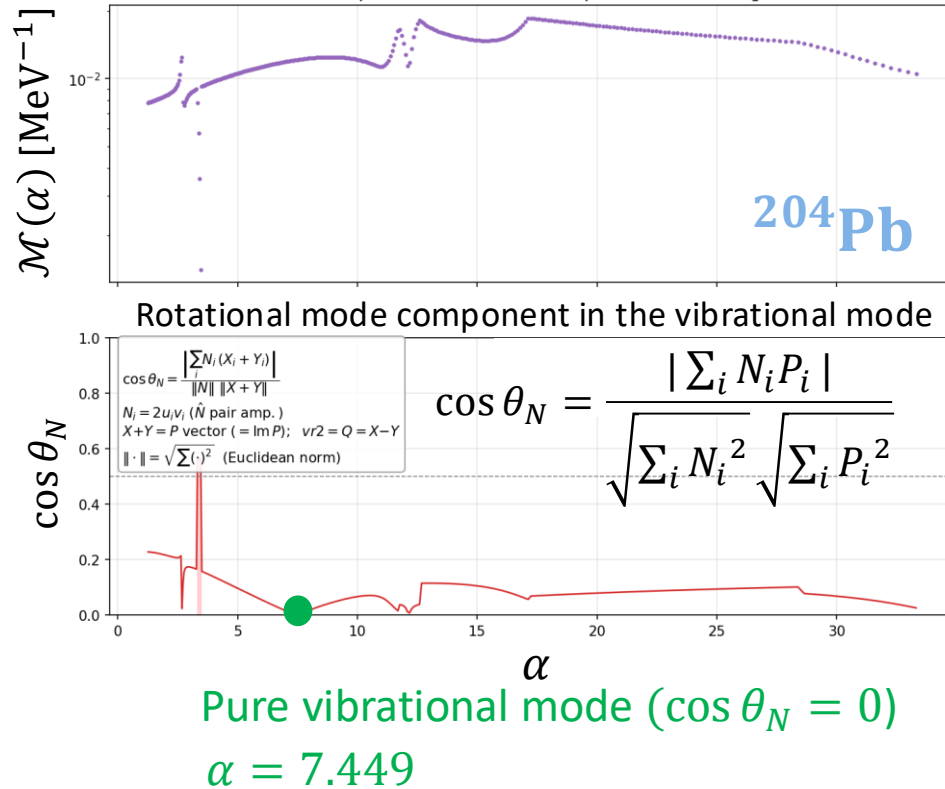
- The QRPA modes are classified into collective modes (with respect to α) and non-collective modes.
- The mass mode (most collective mode for α) switches from mode 24 to mode 9.

The superfluid case : ^{204}Pb



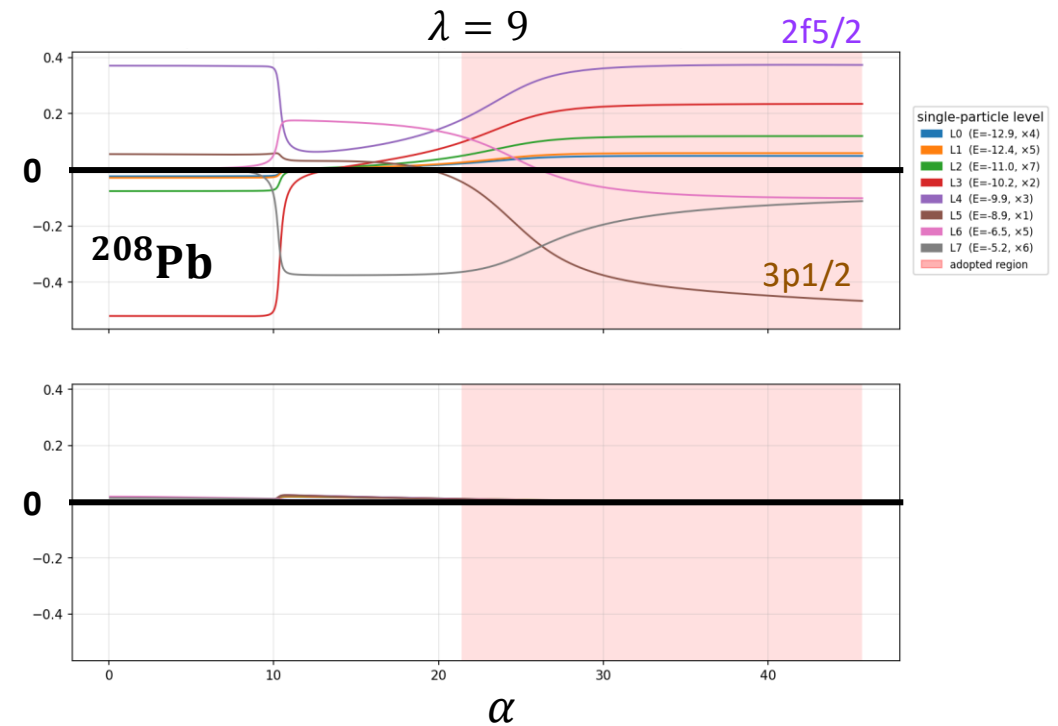
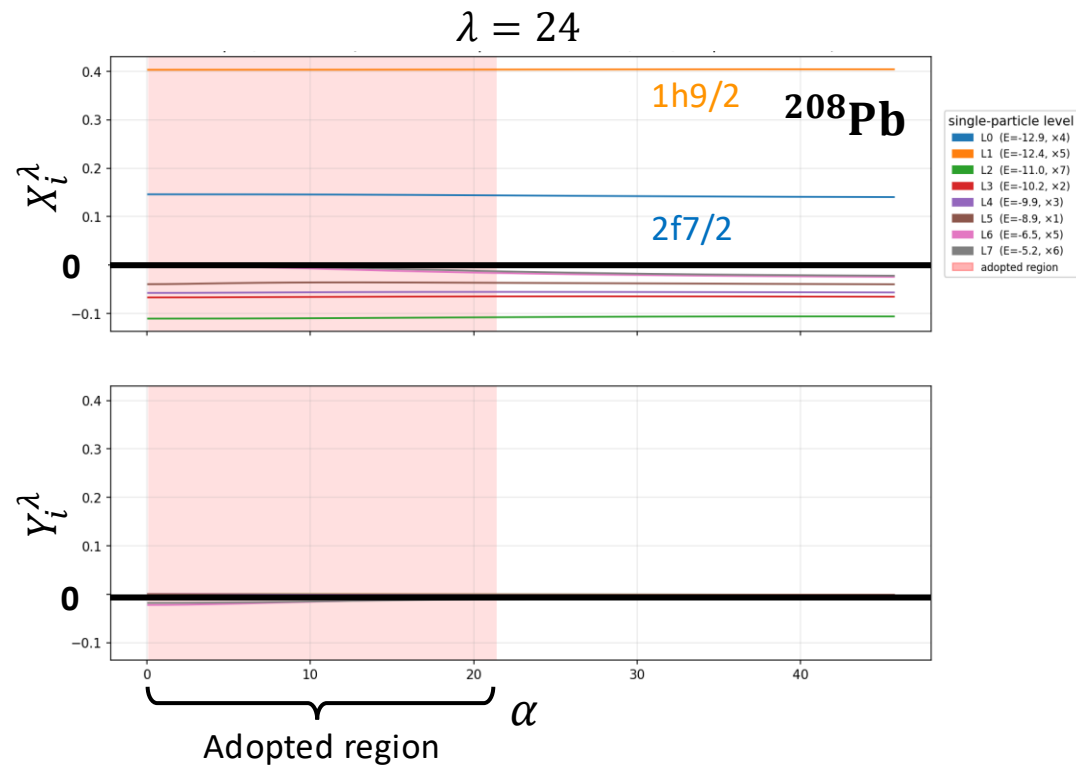
- The pure NG mode can be defined only at the potential minimum of superfluid nuclei, and we have reproduced it numerically.

The superfluid case : ^{204}Pb



- At the potential minimum of the superfluid nucleus ^{204}Pb , we can separate the rotational (pure NG) mode and the vibrational mode.

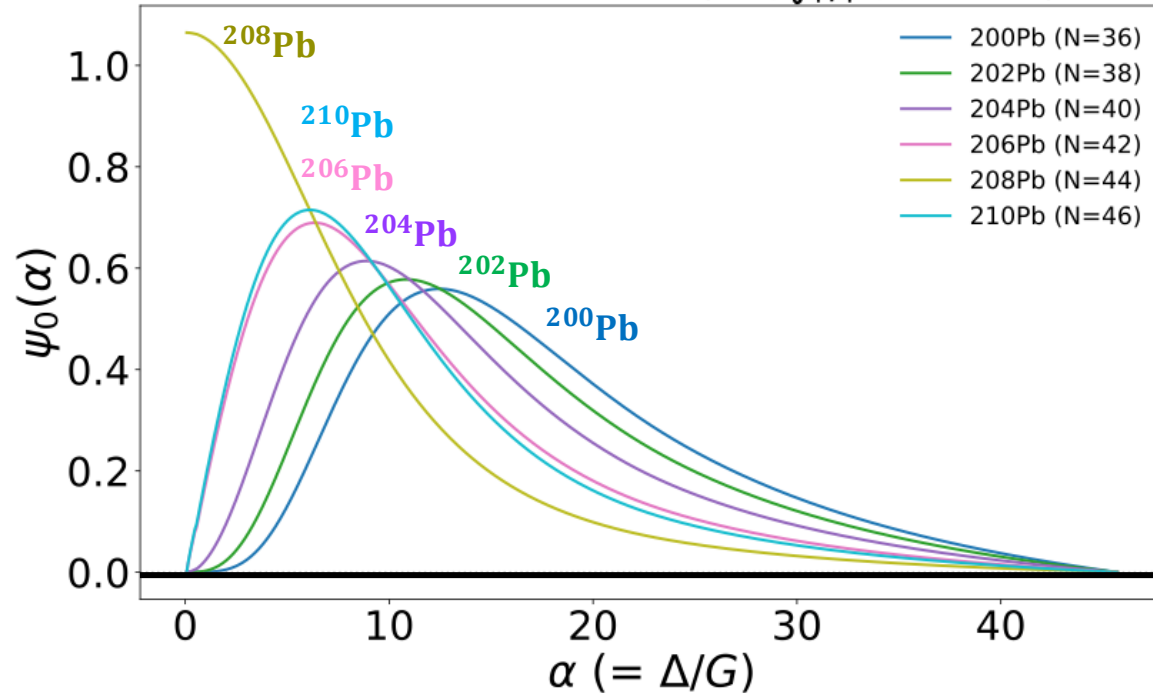
X and Y components of vibrational mode



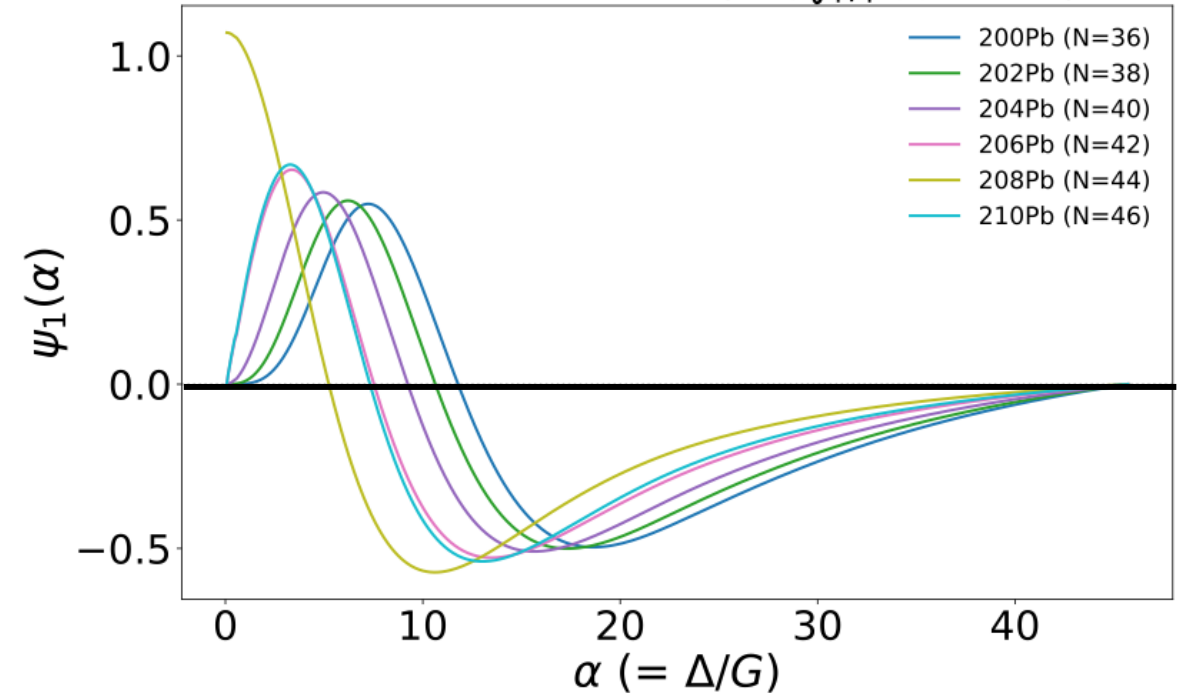
- The selected mass mode changes to a more collective mode at $\alpha \sim 22$.
- 1h9/2 is dominant in mode 24, and 3p1/2 (closest to the Fermi surface) is dominant in mode 9.

Wave functions of PCH

Ground state wavefunctions (raw, $\int |\psi|^2 \sqrt{JB} d\alpha = 1$)

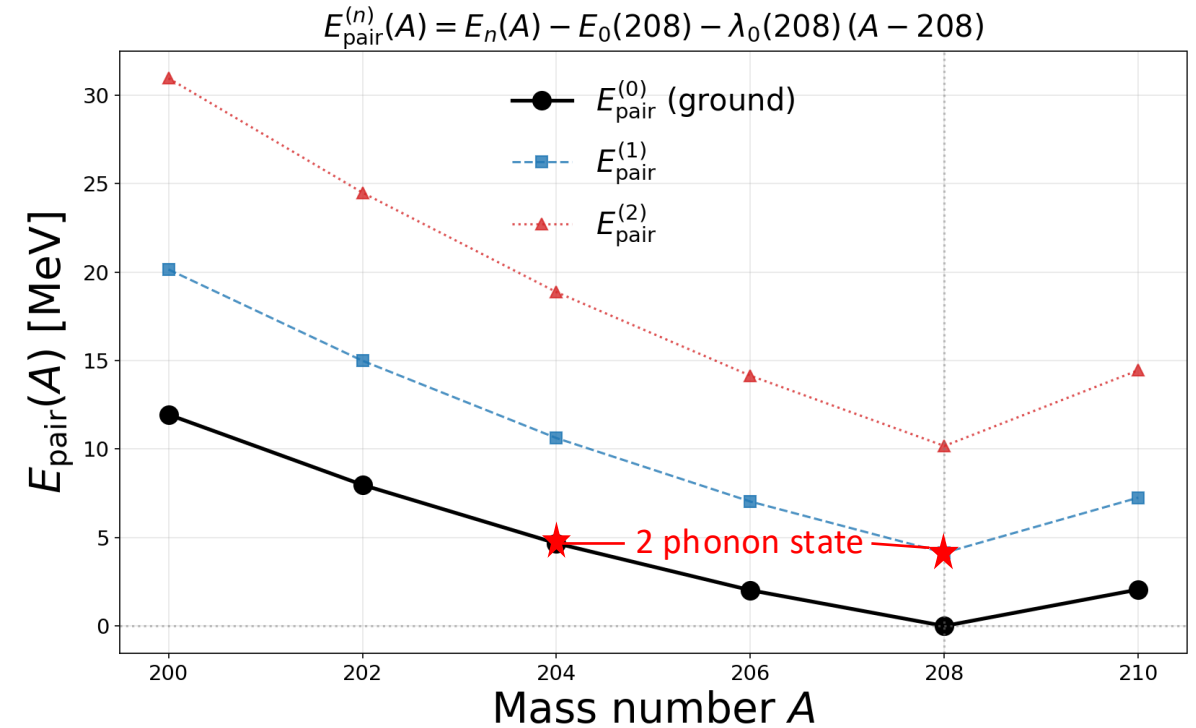
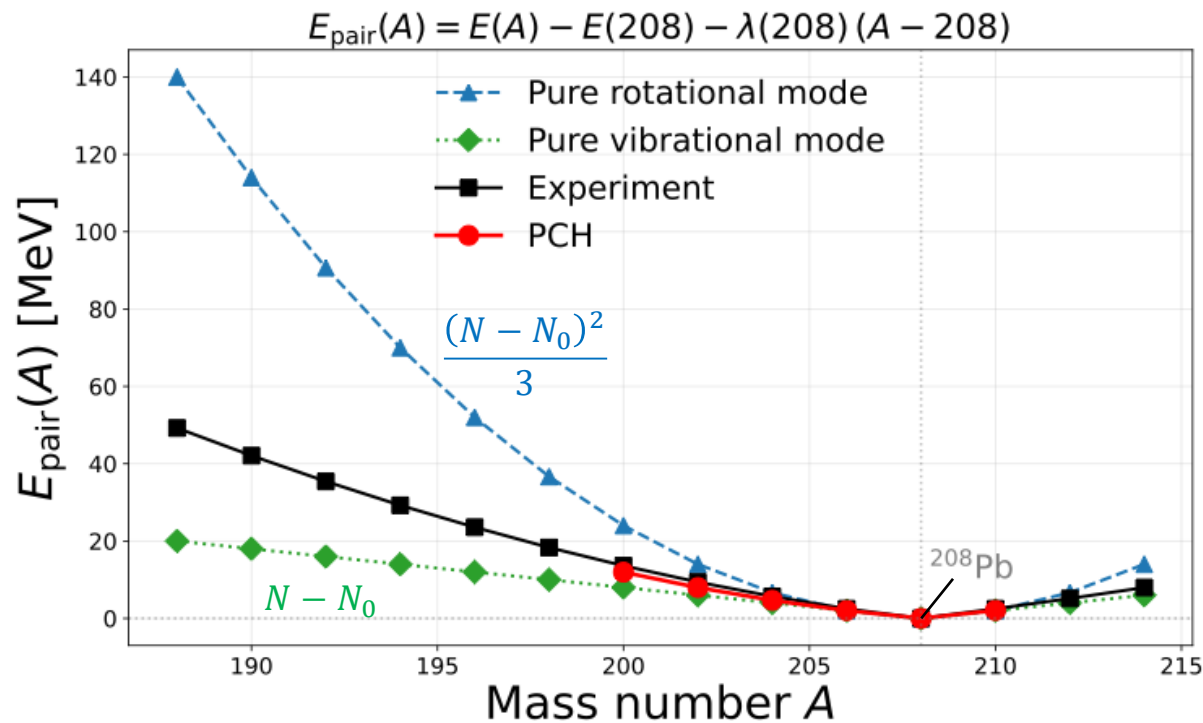


First-excited wavefunctions (raw, $\int |\psi|^2 \sqrt{JB} d\alpha = 1$)



- The wavefunction shifts outward as the nucleus moves away from ^{208}Pb .
- The ground-state wave functions have no node, and the first-excited-state wave functions have one node.

Pairing collective excitation energy



$$E_{\text{pair}}^{(n)}(A) = E_n(A) - E_0(208) - \lambda_0(208)(A - 208)$$

- The PCH reproduces the experimental data of the pairing collective excitation energy E_{pair} .
- Both the experimental data and the PCH results show that the collective energy band around ^{208}Pb lies between the pure vibrational and pure rotational limits.
- There are two kinds of two-phonon states: the ground state of ^{204}Pb and the pairing collective excited state that has the lowest energy of ^{208}Pb . They have almost the same E_{pair} .

Pair transfer intensity

	Expt.	Harmonic	Critical (infinite square well)	Exact	Dynamical PCH (this work)
$^{210}\text{Pb} \rightarrow ^{208}\text{Pb}$					0.98
$^{208}\text{Pb} \rightarrow ^{206}\text{Pb}$	1.00(10)	1.00	1.00	1.00	1.00
$^{206}\text{Pb} \rightarrow ^{204}\text{Pb}$	1.71(17)	2.00	1.53	1.41	1.93
$^{204}\text{Pb} \rightarrow ^{202}\text{Pb}$	2.20(22)	3.00	1.88	1.72	2.80
$^{202}\text{Pb} \rightarrow ^{200}\text{Pb}$					3.58

Solving the PCH analytically using constant J and m for each potential

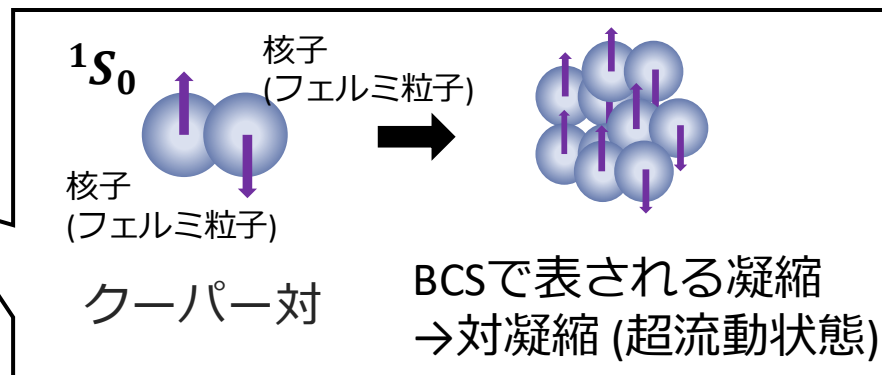
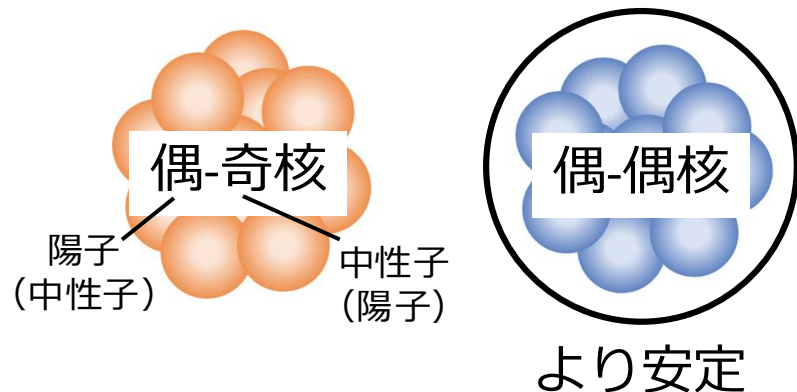
Exact diagonalization of the two-level model Hamiltonian

Clark and Macchiavelli, PRC **77**, 057301 (2008)

$$\text{Normalized pair transfer intensity : } \sigma_{N+2 \rightarrow N} = \frac{\langle \psi_N | \hat{\alpha} | \psi_{N+2} \rangle^2}{\langle \psi_{^{206}\text{Pb}} | \hat{\alpha} | \psi_{^{208}\text{Pb}} \rangle^2}, \quad \hat{\alpha} = \frac{1}{2} \sum_{\mu\nu} (c_{\mu}^{\dagger} c_{\bar{\mu}}^{\dagger} + c_{\bar{\nu}} c_{\nu})$$

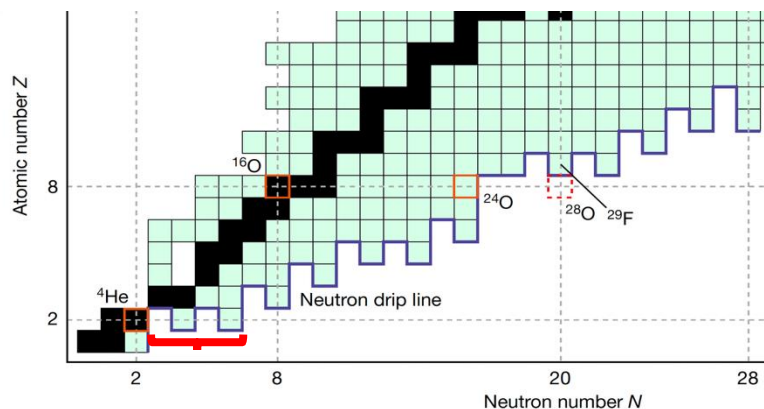
- The PCH roughly reproduces the pair-transfer intensities around ^{208}Pb .
- We can predict the value of pair transfer intensity in the region where no experimental data are available.
- The discrepancy from the experimental values may originate from the values of G and $[\hat{\alpha}, \hat{N}] \neq 0$.

■ 対相互作用



■ 対相関の影響を受ける物理現象

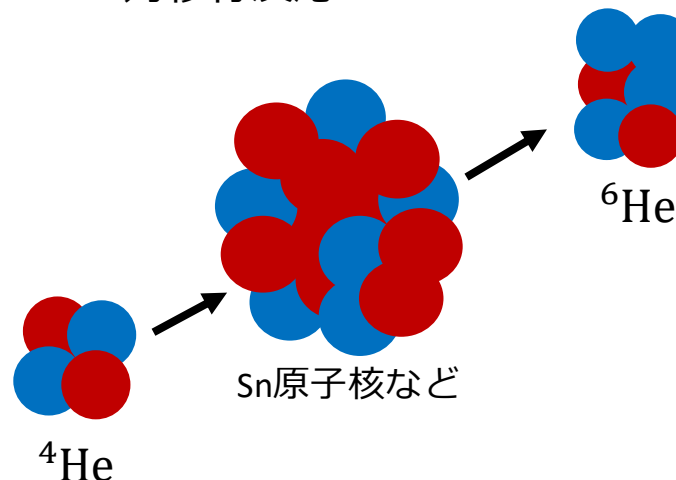
• ドリップライン



Kouno et al. Nature, **620**, p.965–970 (2023)

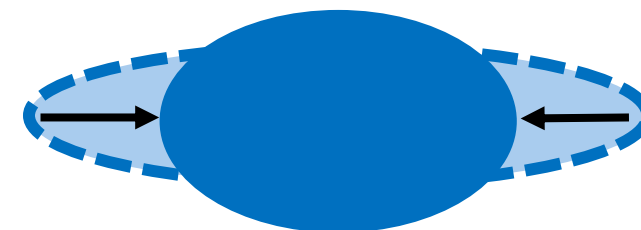
偶数核の方が束縛されやすいため鋸歯状になる

• 対移行反応



中性子2つの対付加等

• 原子核の変形



変形効果の減少

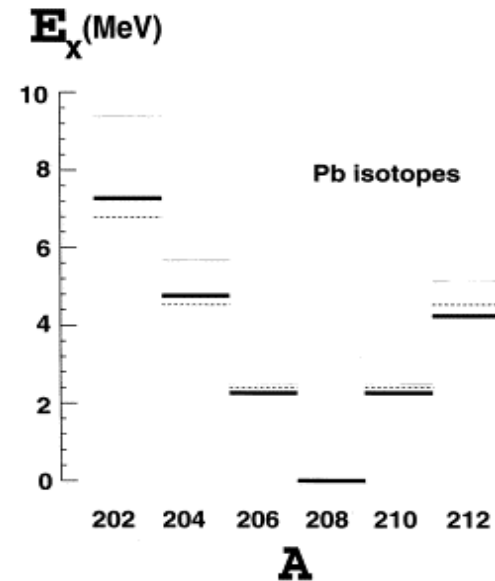
■ 偶-偶核で起こる対相互作用に起因した物理現象

$$E_0(N_0 + \Delta N) = E_0(N_0) + \lambda_n(N_0)\Delta N + E_{\text{pair}}(N_0, \Delta N)$$

基底状態エネルギー フェルミエネルギー 対集団励起エネルギー

魔法数核の周りで発生する
(通常相)

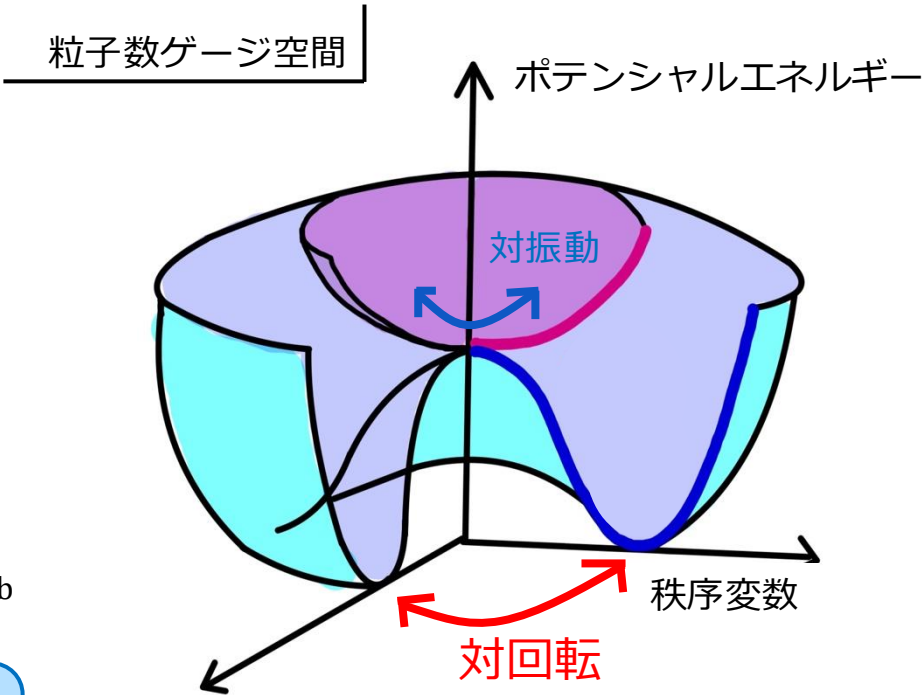
対振動



Pairing collective excitation energy around ^{208}Pb
A.O. Macchiavelli et al. (2000)

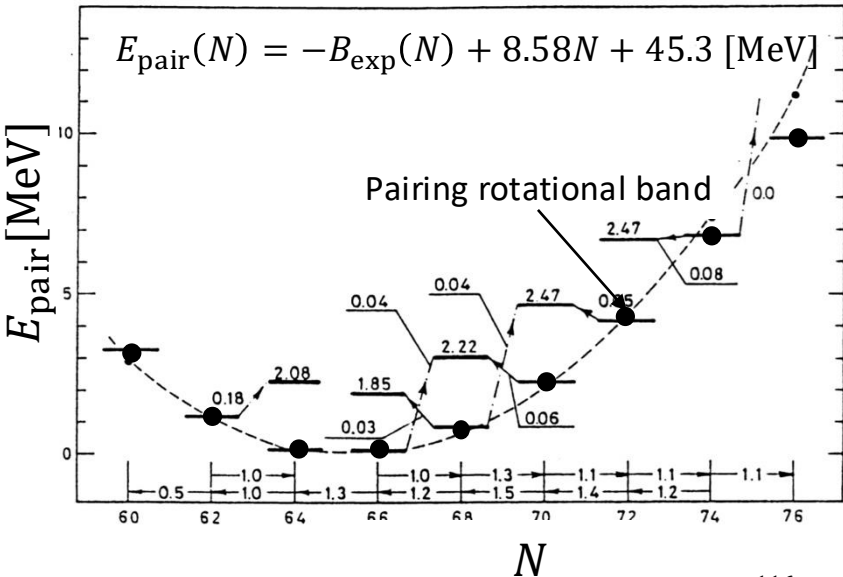
対振動エネルギー $N = N_0 + \Delta N$

$$E_{\text{pair}}^{\text{vib}}(N_0, \Delta N) = \frac{1}{2} \hbar \omega |N - N_0|$$



対回転

開殻の周りで発生する
(超流動相)



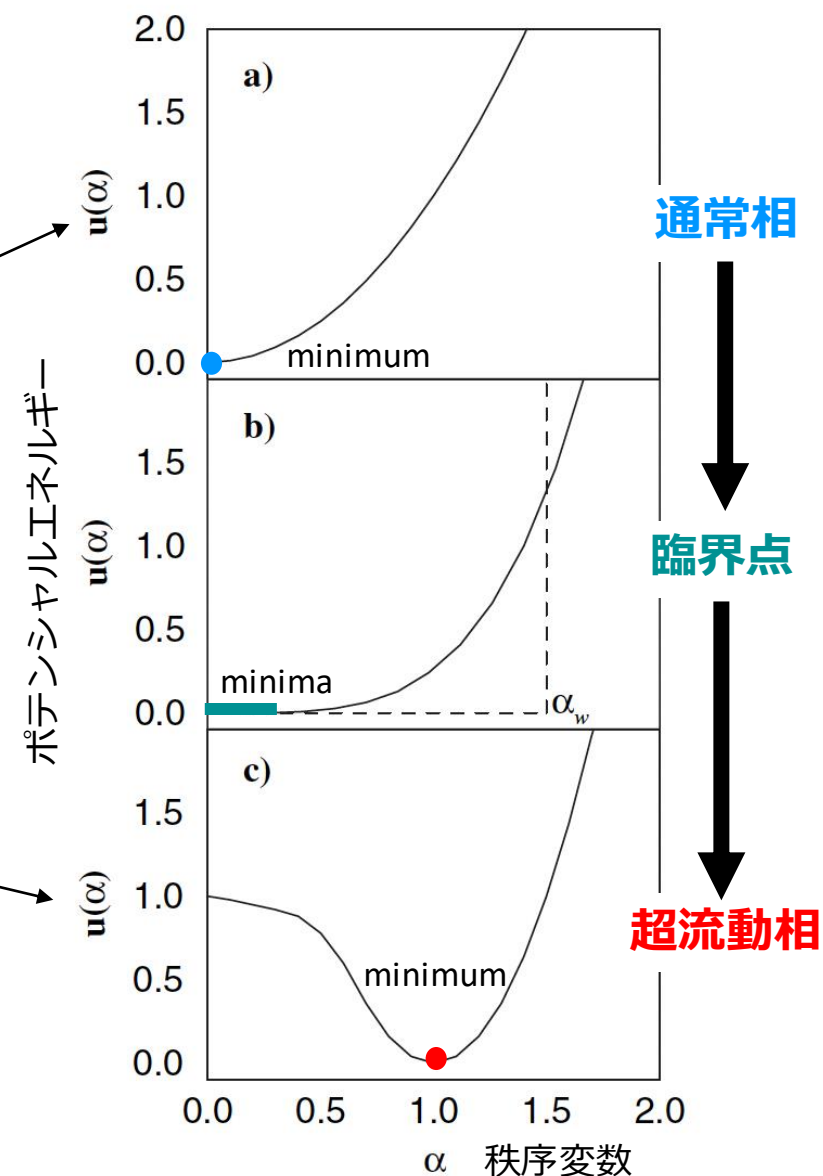
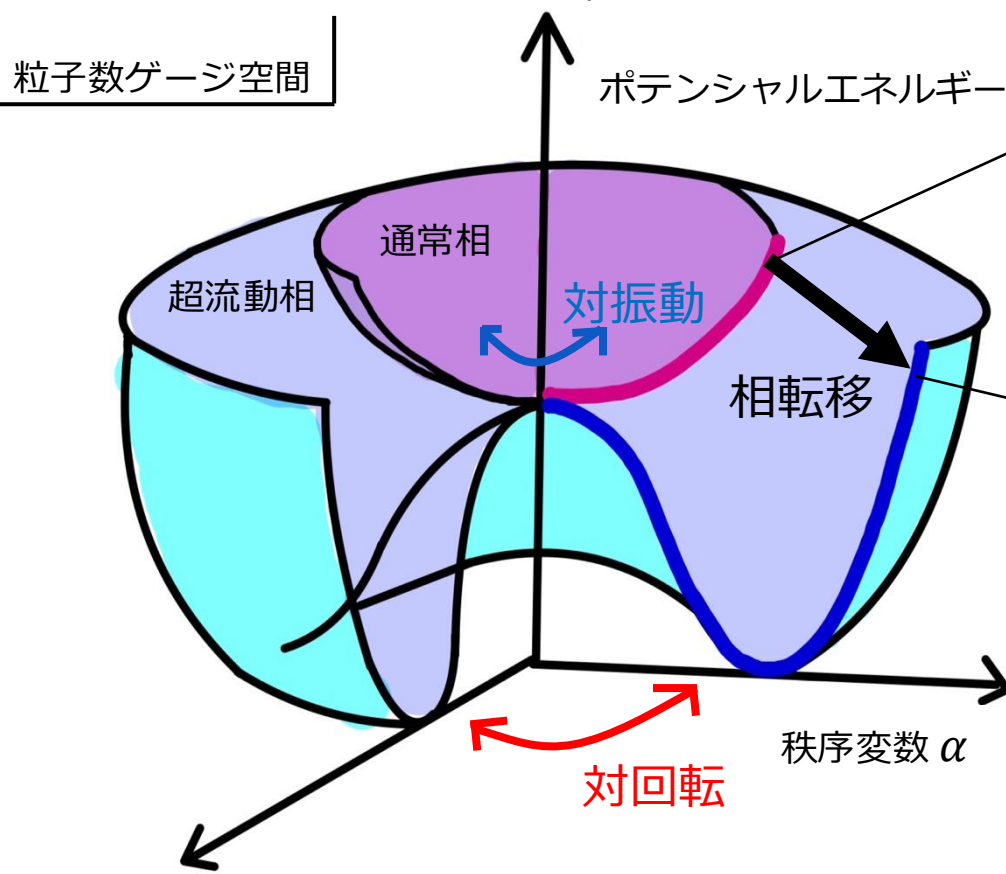
Pairing collective excited energy around ^{116}Sn
Brink and Broglia, Nuclear Superfluidity (2010)

対回転エネルギー $N = N_0 + \Delta N$

$$E_{\text{pair}}^{\text{rot}}(N_0, \Delta N) = \frac{(N - N_0)^2}{2J}$$

臨界点

- 通常相から超流動相への対相転移の中間状態
- ^{208}Pb は魔法数核であるにも関わらず臨界点上にあることが示唆されている (Clark and Macchiavelli, PRC **77**, 057301 (2008))
- Clark et al. (2008)等では現象論的に臨界点を議論しているが微視的な観点からの議論は未だなされていない



- 臨界点も含めた対集団運動の描像を解析するためには、対回転と対振動を同時に記述できる対集団ハミルトニアン¹の構築が不可欠である

$$\hat{H} = \hat{V}(\alpha) + \hat{T}_{\text{rot}}(\alpha) + \hat{T}_{\text{vib}}(\alpha) = V(\alpha) + \frac{\hat{N}^2}{2\mathcal{I}(\alpha)} - \frac{\partial}{\partial \alpha} \frac{1}{2m(\alpha)} \frac{\partial}{\partial \alpha}$$

- 現状の対集団ハミルトニアンはシンプルな分離型の対相互作用を用い、狭い模型空間に限っているため、核図表中の限られた範囲にしか適用できないという問題がある。

(D. R. Bés, R. A. Broglia, R. P. J. Perazzo, and K. Kumar, Nucl. Phys. A, 143, 1-33 (1970))

- 先行研究(D. R. Bés et al. (1970))ではCranking集団慣性質量 $\mathcal{I}(\alpha)$, $m(\alpha)$ を用いており、残留相互作用は考慮されていない。

対集団ハミルトニアン (D. R. Bés et al. (1970))

$$\hat{H}_{\text{coll}} = -\frac{1}{2m(\alpha)} \frac{\partial^2}{\partial \alpha^2} - \frac{1}{4} \left(\frac{1}{\mathcal{I}(\alpha)m(\alpha)} \frac{\partial \mathcal{I}(\alpha)}{\partial \alpha} - \frac{1}{m(\alpha)^2} \frac{\partial m(\alpha)}{\partial \alpha} \right) \frac{\partial}{\partial \alpha} + \left(V(\alpha) + \frac{(N-N_0)^2}{2\mathcal{I}(\alpha)} \right)$$

最終目標

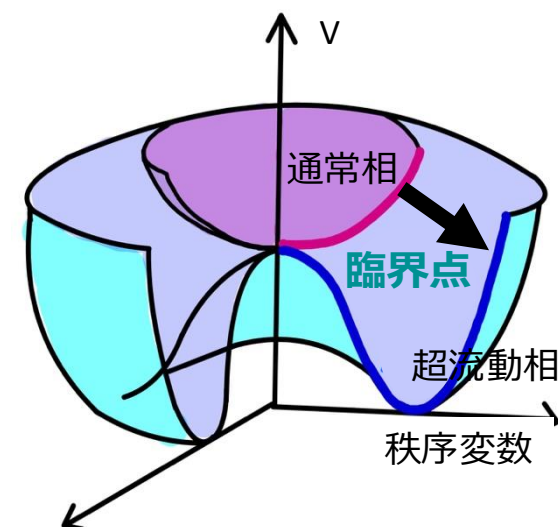
- 相転移臨界点を含めた核図表全域に対して対集団描像を用いて解析するために原子核密度汎関数に基づいた対集団ハミルトニアンを構築する
- 特に魔法核付近の臨界性を検証する

現段階は...

1S_0 中性子対相関のみを考慮した場合の対集団質量とポテンシャルの秩序変数依存性をConstrained BCS + local QRPA 計算を用いて求め、これらを用いた対集団ハミルトニアンを ^{208}Pb 近傍の核に関して解くことで臨界点での対集団モードを調べる

対集団ハミルトニアン

$$\hat{H}_{\text{coll}} = -\frac{1}{2m(\alpha)} \frac{\partial^2}{\partial \alpha^2} - \frac{1}{4} \left(\frac{1}{J(\alpha)m(\alpha)} \frac{\partial J(\alpha)}{\partial \alpha} - \frac{1}{m(\alpha)^2} \frac{\partial m(\alpha)}{\partial \alpha} \right) \frac{\partial}{\partial \alpha} + \left(V(\alpha) + \frac{(N-N_0)^2}{2J(\alpha)} \right)$$



中性子 1S_0 対凝縮秩序変数 α に拘束をかけたBCS計算を行なった

■ 拘束付きBCS計算

秩序変数 α で拘束をかけている

ペアリング項

$$H = \sum_{\nu>0} e_{\nu} (a_{\nu}^{\dagger} a_{\nu} + a_{\bar{\nu}}^{\dagger} a_{\bar{\nu}}) - \frac{G}{2} \sum_{\mu>0} \sum_{\nu>0} (a_{\mu}^{\dagger} a_{\bar{\mu}}^{\dagger} a_{\bar{\nu}} a_{\nu} + a_{\bar{\mu}} a_{\mu} a_{\nu}^{\dagger} a_{\bar{\nu}}^{\dagger})$$

$\bar{\mu}$: μ の時間反転軌道

BCS波動関数を用いて H' の期待値が極小になるように変分計算を行った

$$H' = H - \lambda \hat{N} + C_0 (\hat{\alpha} - \alpha_0)^2$$

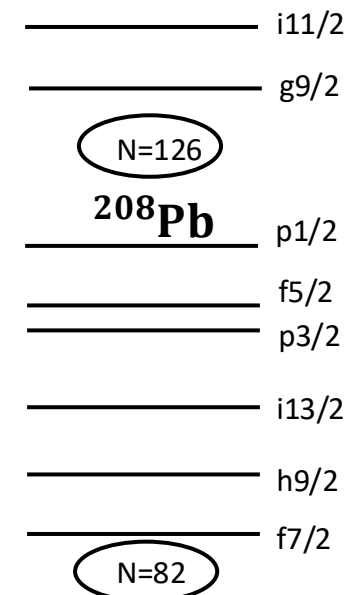
1次線形拘束 2次線形拘束

P.Ring, P.Schuck, The Nuclear Many-Body Problem, Springer

変分を行う上で粒子数 $\langle N \rangle$ と秩序変数 $\frac{\langle a^{\dagger} a^{\dagger} + a a \rangle}{2} = \frac{|\Delta|}{G} = \langle \hat{\alpha} \rangle = \alpha$ に対して拘束条件をかけた
 ペアリング強度は ^{188}Pb のペアリングギャップの実験値でfitした $G \sim 0.16 \text{ MeV}$

➡ポテンシャル曲面 $V(\alpha)$ を得る

Model space (Pb)



■ Thouless-Valatin の対回転慣性モーメント $\mathcal{J}(\alpha)$

$$\mathcal{J}_{\text{TV}} = 2 \sum_{\mu, \nu > 0} N_{\mu\bar{\mu}}^{20} (A + B)_{\mu\bar{\mu}\nu\bar{\nu}}^{-1} N_{\nu\bar{\nu}}^{20}$$

$$\begin{cases} A_{\mu\bar{\mu}\nu\bar{\nu}} = \underbrace{(E_{\mu} + E_{\bar{\mu}})\delta_{\mu\nu}\delta_{\bar{\mu}\bar{\nu}}}_{\text{準粒子エネルギー}} + \underbrace{H_{\mu\bar{\mu}\nu\bar{\nu}}^{22}}_{\text{2準粒子/2空孔成分}} \\ B_{\mu\bar{\mu}\nu\bar{\nu}} = 4! \underbrace{H_{\mu\bar{\mu}\nu\bar{\nu}}^{40}}_{\text{4準粒子成分}} \end{cases}$$

■ 対振動慣性質量 $m(\alpha)$

用いる集団座標における運動エネルギー項の係数

$$T = \frac{1}{2} \sum_i \dot{q}_i = \frac{1}{2} \sum_{mn} m_{mn}(\alpha) \dot{\alpha}_m \dot{\alpha}_n$$

$$m(\alpha) = \left(\langle \text{CBCS}(\alpha) \left| \left[\hat{\alpha}, \frac{1}{i} \hat{P}(\alpha) \right] \right| \text{CBCS}(\alpha) \rangle \right)^{-2}$$

P,Q表示のQRPA計算における演算子 $\hat{P}$

➡ここまでに求めた $V(\alpha)$, $\mathcal{J}(\alpha)$, $m(\alpha)$ を用いて対集団ハミルトニアンを解く

$$\hat{H}_{\text{coll}} = -\frac{1}{2m(\alpha)} \frac{\partial^2}{\partial \alpha^2} - \frac{1}{4} \left(\frac{1}{\mathcal{J}(\alpha)m(\alpha)} \frac{\partial \mathcal{J}(\alpha)}{\partial \alpha} - \frac{1}{m(\alpha)^2} \frac{\partial m(\alpha)}{\partial \alpha} \right) \frac{\partial}{\partial \alpha} + \left(V(\alpha) + \frac{(N-N_0)^2}{2\mathcal{J}(\alpha)} \right)$$

Q&A

- BESのやり方とは何が違う？

使っているハミルトニアンとポテンシャルの求め方は同じBCS。ただmoiとmassの求め方がBESはcrankingなのに対し私はQRPAを用いている。Residual interactionを含めるかどうかの差。Residual interactionは少なからず重要であることがmoiの結果からもわかる。微分経路とも関係する。

- Collective massが1/MeVなのはなぜ

$$T = \frac{1}{2} \sum_i \dot{q}_i^2 = \frac{1}{2} \sum_{mn} \mathcal{M}_{mn}(s) \dot{s}_m \dot{s}_n$$

次元解析をすると

$$[\text{energy}] = [M] \left[\frac{1}{\text{time}} \right]^2$$

自然単位系 $\hbar = 1$ とすると $[\text{energy}] = \left[\frac{1}{\text{time}} \right]$ より

$$[M] = \left[\frac{1}{\text{energy}} \right] = [\text{MeV}]^{-1}$$

Q&A

- 1番小さいmassがpairing vibrational modeのmassである根拠は？

松尾さんのHiggs modeの論文では、QRPA方程式を解いた後に各モードに対してpair-additional/removal reactionの遷移強度を計算し、それが1番大きいモードで計算されるmassをvibration massとして採用している。

→つまり1番対移行しやすいモードがvibrational modeであるとしている

→妥当な気もするが、では他のQRPAモードはどのような励起を表しているのか？最も対移行しやすいというのがvibrational modeの定義なのだろうか？

日野原さんの2010年の論文

ASCC(Adiabatic-Self-consistent-Collective-Coordinate)はoriginalな集団座標 q を変分によって決める理論。これにmassという物理量を導入して別の簡単な集団座標 s に移行し、最適な（最も集団エネルギーの低い？）ものを求めるのがCHFB+LQRPA。よって集団エネルギー（QRPAの固有値）のうち最も小さな2つのモードを選べばそれらは最も適切な集団座標 s を持つ（最も集団性の高い）モードである保証がある？

集団エネルギーが小さいモードはmassも小さいからmassが最小なモードをとっている？

ASCCでは集団座標 q を少しずつ変えながらLQRPA計算を行い、HFB解が不安定になったら終了する。この過程で前のモードと最もoverlapの大きいモードを探しておき、固有値が小さい中で最も連続性の高いモードを用いてmassを計算する

- order parameterの虚軸方向はどう含めるのか？

→BESの論文にgapの虚軸方向成分も含めて書いてあった気がする

Q&A

- order parameterの虚軸方向はどう含めるのか？

→BESの論文

物体固定系ではorder parameter, P の期待値はrealなので虚軸方向成分は0としている。

つまりこれはpairing変形の変形軸の方向をreal軸と一致させるように固定するというgauge fixをしている？

ただこれではどのように回転を考えるのか？ Intrinsic systemのまま回転を考えれば良いのか？

Q&A

■ 量子化の仕方

パウリの量子化の方法で、係数をまとめて簡略化したもの

■ masslessモードに関して

massless（エネルギーがいない回転モード）と言っているのになぜ対回転エネルギーがあるのか

＞ 超流動相転移に伴う大域的 $U(1)$ 対称性の破れ(BCS/HFB状態をゲージ回転してもエネルギーが変わらない)によって発生するNGモードは正確にはエネルギーの「ほとんど」いない低励起回転モードだから多少はmassがある？（そもそも $I=0$ だと回転エネルギーを定義できないため）

またはmasslessと表現しているのは物体固定系におけるゲージ回転（ $\exp(iN\phi)$ かける操作）においてエネルギー0で回れると言っているのもであって、それを実験室系で見たら（つまり粒子数を決めて球形になったら）回転エネルギーが見えるということでは？

NGボソンが場とカップルして質量を持つことはないのか

＞ 超流動相転移で破れた対称性はグローバル $U(1)$ 対称性であり、これは局所対称性を示すゲージ対称性ではないため、NGボソンがゲージ場の縦波成分となりゲージ場が質量をもつヒッグス機構は起きないのではないかと？
→やはり粒子数「ゲージ」空間というネーミングはナンセンスでは？

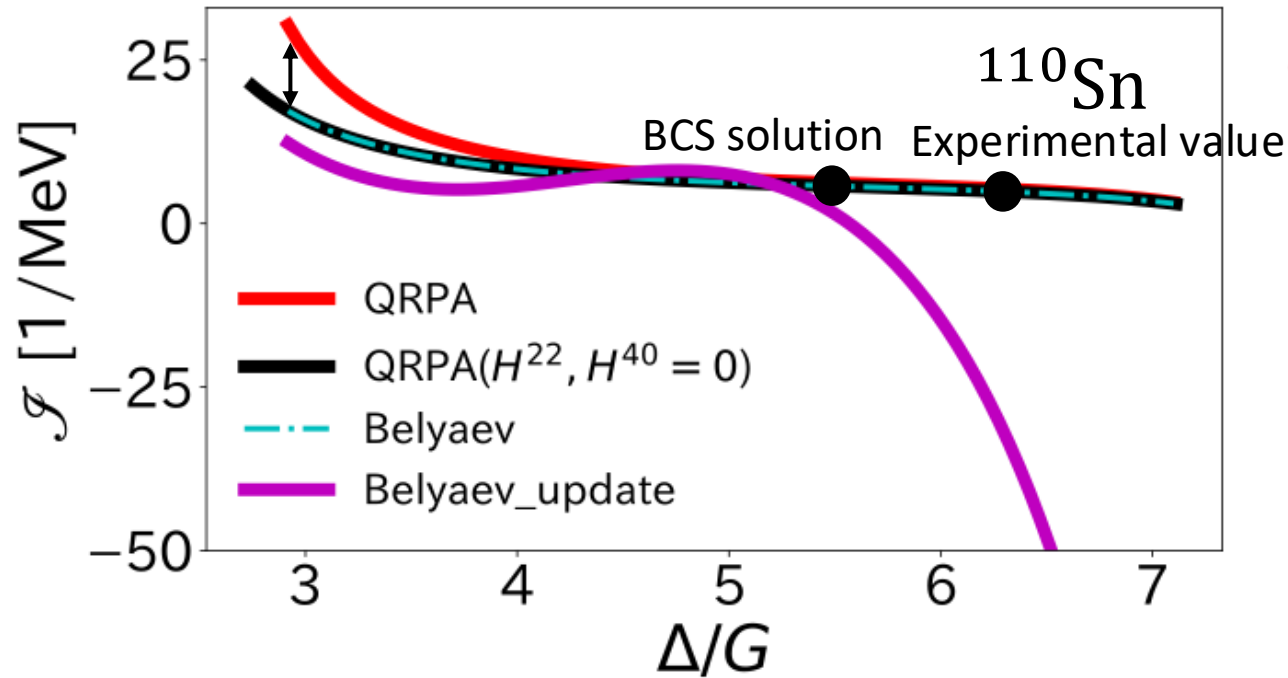
ωJx は剛体仮定だけどHに流動性があるから？

High-spinになると内部スピンの向きが揃うから1S0超流動は組めなくなる

■ ThoulessValatinの慣性モーメントはどうやって求められたのか

＞ まず原子核の慣性モーメントは高スピン状態などで超流動性を失った剛体と考えられる時の I_{rigid} と超流動性がある低励起状態での $I_{\text{irrotational}}$ がある。クランキング公式は $H' = H - \omega Jx$ として求めた波動関数から得られる $\langle Jx \rangle$ を用いて $I = \langle Jx \rangle / \omega$ で求められるものであり、rigidとirrの間を与える。Cranking慣性モーメントでもペアリングを入れないのがInglis, 入れたのがBelyaevである。さらに粒子-空孔相関などを取り入れて線形応答理論から求めたものがThouless-Valatinの慣性モーメント。

Supplement: λ - Δ - μ plane (P-MOI)



CBCS + LQRPA

CBCS + LQRPA
($H^{22}, H^{40} = 0$)

CBCS + Belyaev
(Partial derivative)

CBCS + Belyaev_update
(Total derivative)

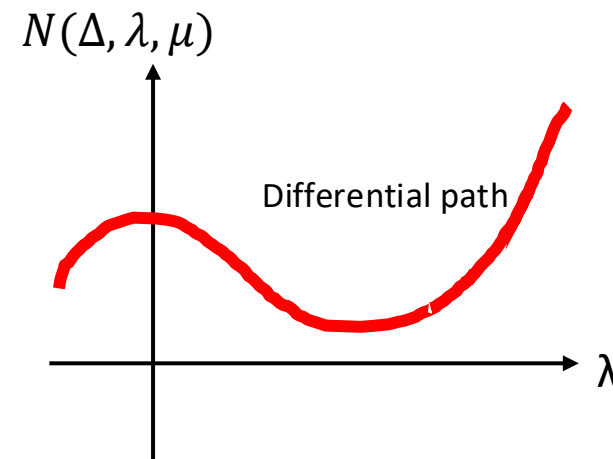
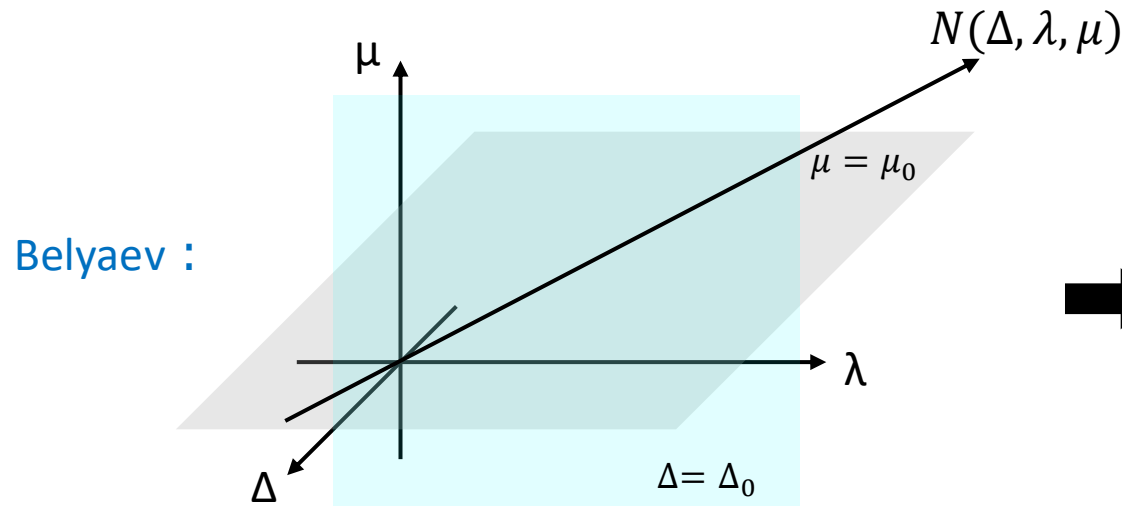
$$\mathcal{J}_{TV} = 2 \sum_{\mu, \nu > 0} N_{\mu\bar{\mu}}^{20} (A + B)_{\mu\bar{\mu}\nu\bar{\nu}}^{-1} N_{\nu\bar{\nu}}^{20}$$

✱ assuming $H^{22}, H^{40} = 0$ in A, B

$$\mathcal{J}_{\text{Belyaev}}(N) = \frac{\partial N(\Delta, \lambda, \mu)}{\partial \lambda} \Big|_{\Delta, \mu} = 4 \sum_{\mu > 0} \frac{U_{\mu} V_{\mu}}{E_{\mu}}$$

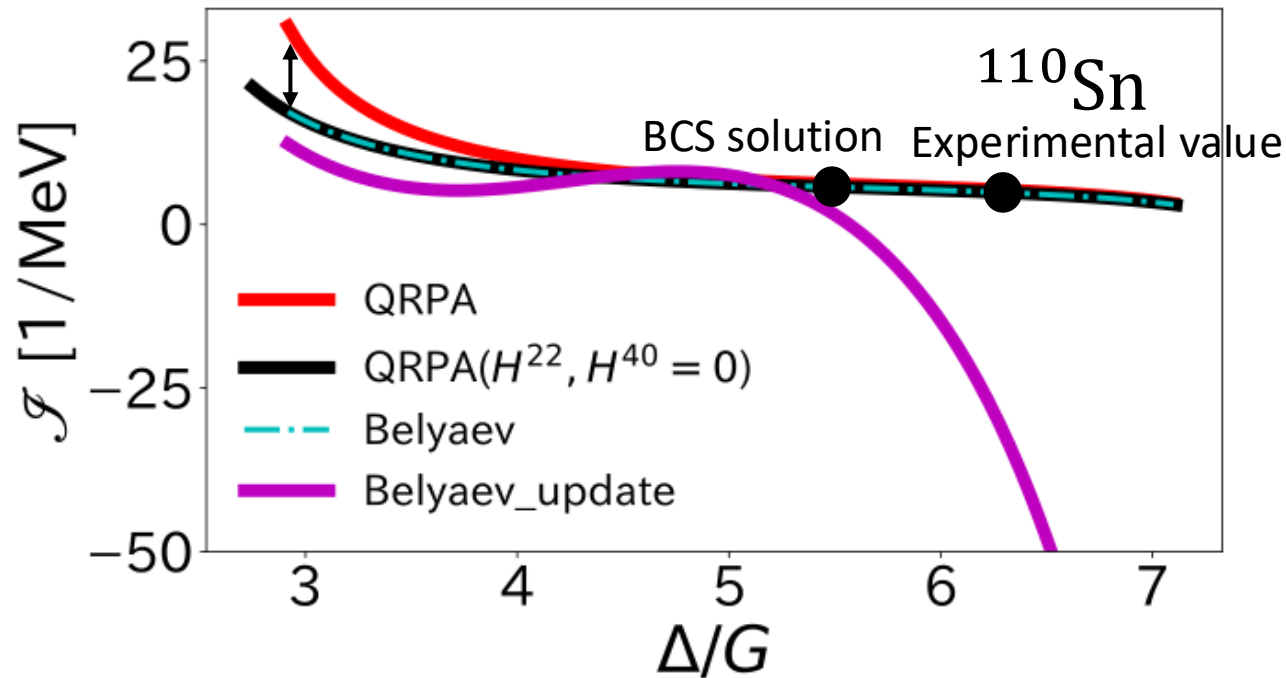
$$\mathcal{J}_{\text{Belyaev_update}}(N) = \frac{dN(\Delta, \lambda, \mu)}{d\lambda} = \frac{\partial N}{\partial \lambda} + \frac{\partial N}{\partial \Delta} \frac{\partial \Delta}{\partial \lambda} + \frac{\partial N}{\partial \mu} \frac{\partial \mu}{\partial \lambda}$$

Residual interaction



We must follow the differential path across the entire λ - Δ - μ plane.

Supplement: λ - Δ - μ plane (P-MOI)



CBCS + LQRPA

CBCS + LQRPA
($H^{22}, H^{40} = 0$)

CBCS + Belyaev
(Partial derivative)

CBCS + Belyaev_update
(Total derivative)

$$\mathcal{J}_{TV} = 2 \sum_{\mu, \nu > 0} N_{\mu\bar{\mu}}^{20} (A + B)_{\mu\bar{\mu}\nu\bar{\nu}}^{-1} N_{\nu\bar{\nu}}^{20}$$

✱ assuming $H^{22}, H^{40} = 0$ in A, B

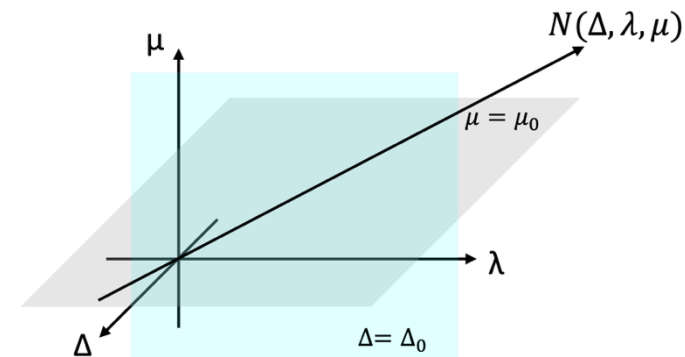
$$\mathcal{J}_{\text{Belyaev}}(N) = \frac{\partial N(\Delta, \lambda, \mu)}{\partial \lambda} \Big|_{\Delta, \mu} = 4 \sum_{\mu > 0} \frac{U_{\mu} V_{\mu}}{E_{\mu}}$$

$$\mathcal{J}_{\text{Belyaev_update}}(N) = \frac{dN(\Delta, \lambda, \mu)}{d\lambda} = \frac{\partial N}{\partial \lambda} + \frac{\partial N}{\partial \Delta} \frac{\partial \Delta}{\partial \lambda} + \frac{\partial N}{\partial \mu} \frac{\partial \mu}{\partial \lambda}$$

Residual interaction

- $\mathcal{J}_{\text{Belyaev_update}}$ should match the LQRPA solution, but due to the problem of my code, it does not match.

The difference between the results of Thouless-Valatin and Belyaev arises from the fact that they follow **different differential paths**.



P-MOI

大域的U(1)対称性の破れ

BCS状態：粒子数ゲージ空間において特定の方向をもち、さまざまな粒子数状態の重ね合わせで表現されているため、粒子数演算子の固有状態ではなく、BCS近似されたハミルトニアンは粒子数演算子と交換しなくなる。

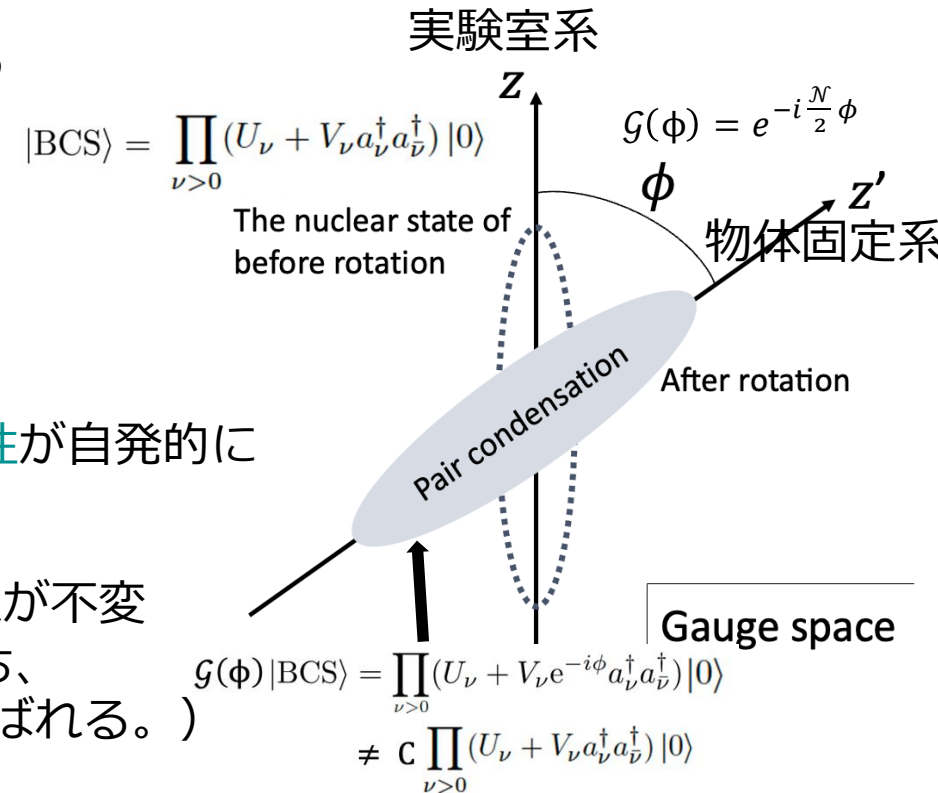
BCS波動関数： $|\text{BCS}\rangle = \prod_{\nu>0} (U_\nu + V_\nu \overbrace{a_\nu^\dagger a_{\bar{\nu}}^\dagger}^{\text{様々な粒子数状態の重ね合わせ}}) |0\rangle$

実験室系での極座標系において対称軸z軸からの角度である天頂角 $\theta = \phi$ 回転させた時（物体固定系の方向z'とする）のBCS状態：

$$\begin{aligned} \mathcal{G}(\phi) |\text{BCS}\rangle &= \prod_{\nu>0} (U_\nu + V_\nu e^{-i\phi} a_\nu^\dagger a_{\bar{\nu}}^\dagger) |0\rangle \quad \mathcal{G}(\phi) = e^{-i\frac{N}{2}\phi} \\ &\neq c \prod_{\nu>0} (U_\nu + V_\nu a_\nu^\dagger a_{\bar{\nu}}^\dagger) |0\rangle \end{aligned}$$

この状態は元のBCS状態とは異なる状態になっており、大域的U(1)対称性が自発的に破れていると言える。

大域的U(1)対称性：ある量子力学的状態に対して位相 $e^{i\theta}$ をかけても状態が不変である性質。（位相 $e^{i\theta}$ の集合は結合法則が成り立ち、単位元と逆元が存在するため群をなし、U(1)群と呼ばれる。）



粒子数ゲージ空間：回転演算子が $g = e^{iN/2\phi}$ で表される空間。

BCS近似：ペアリングハミルトニアン H_{00}, H_{11} の準粒子表示での行列要素を H_{00}, H_{11} のみ残し、 H_{20} 以降は0とする近似。

数の集合が群をなす条件：

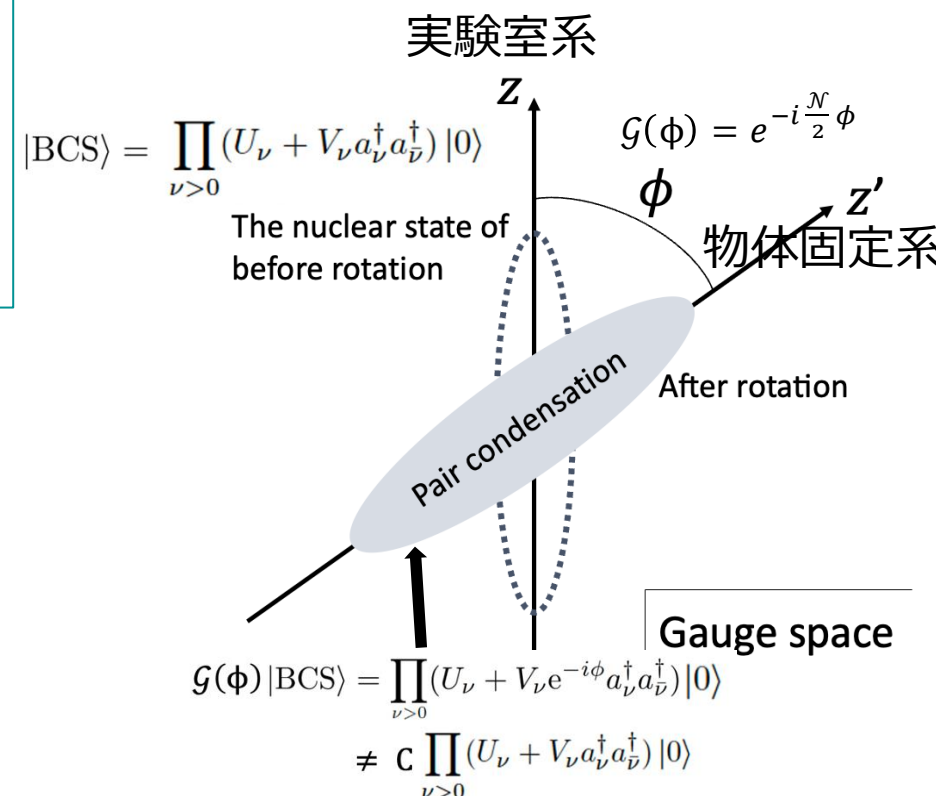
✓ 結合法則 $x, y, z \in G$ に対し、 $x * (y * z) = (x * y) * z$ が成り立つ。

✓ 単位元 $x * e = e * x = x$ を満たす $e \in G$ が存在する ($x \in G$)。

✓ 逆元 $x * y = y * x = e$ を満たす $y \in G$ が存在する ($x \in G$)。

✓ 可換性 $x * y = y * x = e$ を満たす $x, y \in G$ が存在する。

※可換性まで満たす群はアーベル群と呼ばれる



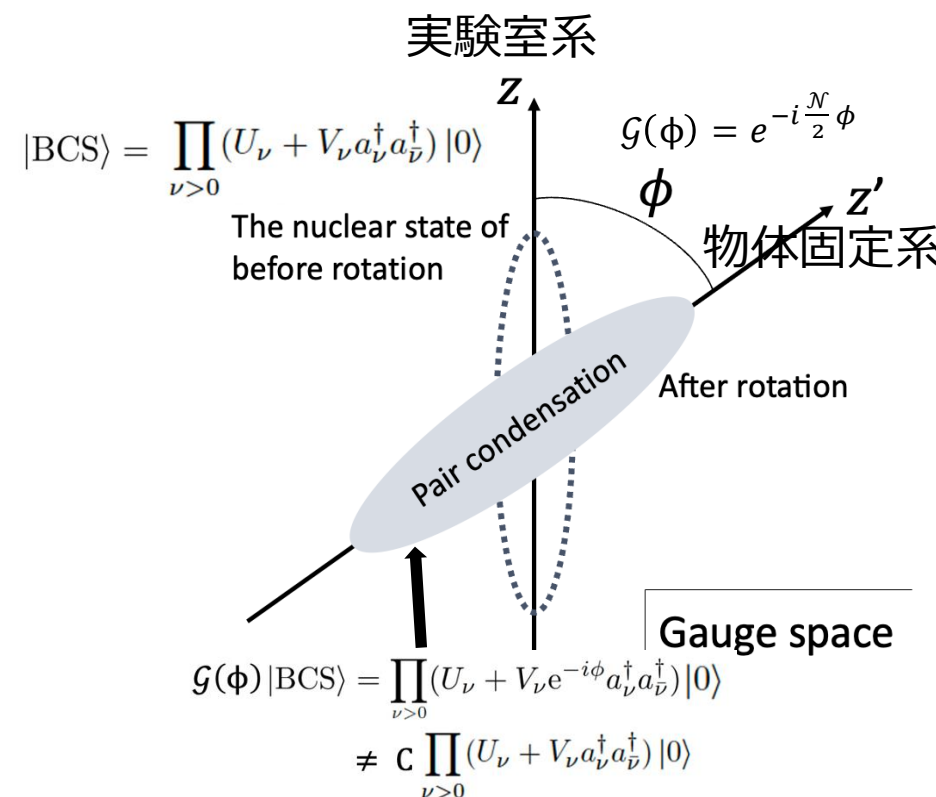
大域的U(1)対称性の破れ

本研究で用いている粒子数ゲージ空間の回転演算子 $g = e^{iN/2\phi}$ 位相角 ϕ は位置や時間を変数にもたないため、この回転演算子（位相）をかける操作に対して不変であることを大域的U(1)対称性があるという。一方 $e^{iN/2\phi(x,t)}$ のように位置や時間に依存した位相をかける操作に対して不変であることをU(1)ゲージ対称性または局所U(1)対称性があるという。

ゲージ対称性：四元ポテンシャル A_μ に対してゲージ変換 $A_\mu \rightarrow A_\mu - \partial_\mu \chi$ ($\chi(x,t)$ はスカラー関数) を行っても不変であること。

波動関数に対するゲージ変換は $e^{iN/2\phi(x,t)}$ を作用させること??
なので位置と時間に依存しない $e^{iN/2\phi}$ を作用させる操作はゲージ変換ではない??

なぜ粒子数ゲージ空間と呼ぶ?



実空間での（通常の）原子核の**回転**

一般化座標 q と一般化運動量 p を共役な正準変数として定義

$$[q, p] = i \quad \text{※ } \hbar = 1$$

ハミルトニアンを用いた慣性質量 m の定義は

$$\frac{\partial^2 H}{\partial p^2} = \frac{1}{m}$$

+ 物理的考察から

これらを適切に正準変換（交換関係が同型である変換）すると、軸対称変形した原子核の、回転運動における回転角 φ と角運動量演算子 L は $q, p \rightarrow \varphi, L$ と対応する共役変数であることがわかる

$$[\varphi, L] = i$$

回転運動の質量とみなせる慣性モーメントもハミルトニアンから同様に定義できる

$$\frac{\partial^2 H}{\partial L^2} = \frac{1}{I}$$

(Second-MOI)

対回転（ゲージ空間内での原子核の回転）

ゲージ角 ϕ と粒子数演算子 N で同様のことが言える

これらを適切に正準変換すると、ゲージ角 ϕ と粒子数演算子 N は回転運動とのアナロジーから $q, p \rightarrow \phi, N$ と対応する共役変数であることがわかる

$$[\phi, N] = i$$

対回転運動の質量とみなせる対回転の慣性モーメントもハミルトニアンから同様に定義できると考えられるので

$$\frac{\partial^2 H}{\partial N^2} = \frac{1}{J}$$

実空間での（通常の）原子核の**回転**

$$\frac{\partial^2 H}{\partial L^2} = \frac{1}{I}$$

対回転（ゲージ空間内での原子核の回転）

$$\frac{\partial^2 H}{\partial N^2} = \frac{1}{J}$$

よって H の期待値 E_0 を粒子数 N で展開すると

$$\begin{aligned} E_0(N) &= E_0(N_0) + \frac{\partial E_0}{\partial N} (N - N_0) + \frac{1}{2} \frac{\partial^2 E_0}{\partial N^2} (N - N_0)^2 \\ &= E_0(N_0) + \frac{\partial E_0}{\partial N} (N - N_0) + \frac{(N - N_0)^2}{2J} \end{aligned}$$

- ペアリングギャップ(理論値) $\neq$ 偶奇質量差 (実験値)
- それぞれに多様な定義がある

ペアリングギャップ (理論値)

$$\Delta = G \sum_{\nu} \Omega_{\nu} U_{\nu} V_{\nu}$$
$$\vdots$$

偶奇質量差, OES (実験値)

$$\Delta_n^{(3)}(N) = \frac{(-1)^N}{2} [E(N+1) - 2E(N) + E(N-1)],$$
$$\vdots$$

ペアリングギャップとOESが相当すると言われている原因は何か？

ペアリングを考えずに核の質量を計算すると、偶数核と奇数核で系統的なエネルギー差は見られないが、ペアリングの項をハミルトニアンに追加すると偶数核の基底エネルギーが系統的に低くなる。

その偶数核と隣の奇数核の主なエネルギー差は理論の定義でのペアリングギャップなので、これがOESに相当していると言っているだろう、という考え方。

ほんと？BCSでも？

偶奇質量差

$$\Delta^{(1)}(N) = E(N) - E(N-1)$$

3点公式

$$\Delta^{(3)}(N) = \frac{\Delta(N+1) - \Delta(N)}{2} = \frac{1}{2} [E(N+1) - E(N) - E(N) + E(N-1)] = \frac{E(N+1) - 2E(N) + E(N-1)}{2}$$

4点公式

5点公式

・
・
・

※3点,4点,5点...公式でもやりたいことは $\Delta^{(1)}(N)$ の平均値をとってより正確な(局所的な性質を拾わない) OESを求めたい

エネルギーの微分を行いたいというモチベーションではなく、
ヤナックの定理 (クーブマンズの定理) とは異なる

ヤナックの定理 密度汎関数法で成り立つ定理。

全エネルギー

$$\frac{\partial E_{tot}}{\partial f_i} = \varepsilon_i$$

各軌道のエネルギー

各軌道iの占有率

核図表の広い範囲を計算できる密度汎関数を用いた奇数核の基底状態の計算の大半は近似的に計算している

- ✓ 軸対称変形
- ✓ Time-odd場を無視 奇数核は偶数のペア+1核子で構成されるため、時間反転対称性の破れた項が存在する
etc....

→このような近似や仮定をおくと、1粒子エネルギーの値があまり精度の良くないものになってしまう

M. Zalewski, J. Dobaczewski, W. Satuła, and T. R. Werner, Phys. Rev. C 77, 024316 (2008).

M. Kortelainen, J. Dobaczewski, K. Mizuyama, and J. Toivanen, Phys. Rev. C 77, 064307 (2008).

→このような近似や仮定が計算結果にどのような影響を与えるのか調べる必要がある

Schuck et al. PRC 81,02436, (2010)

密度汎関数法 (HF,HFB method)

密度汎関数法：多体系のハミルトニアンを粒子密度の関数として表す理論。

Hartree-Fock(HF)法：多体系の波動関数をSlater行列式であると仮定し、多体相互作用を1体相互作用に近似することで多体のSchrödinger方程式を1体のSchrödinger方程式に落とし込む理論。

Hartree-Fock-Bogoliubov(HFB)法：Bogoliubov変換によって準粒子という概念を取り入れたHF法の拡張理論。対相関を含めた1体場を構築できる。

※原子核理論分野では密度汎関数法とHF,HFB法はほぼ同義として使用されている

- ① ✓ 有効相互作用の種類 (Skyrme, Gogny, Fayans)
 ✓ N体系の波動関数をSlater行列式で近似し、これを構成する1粒子波動関数を展開する基底 (調和振動子, 座標)
 ✓ ユニタリ行列 $W(\mathbf{U}, \mathbf{V})$ (Bogoliubov変換の変換行列)の初期値を決める

$$\begin{pmatrix} \alpha \\ \alpha^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} U^\dagger & V^\dagger \\ V^T & U^T \end{pmatrix}}_W \begin{pmatrix} a \\ a^\dagger \end{pmatrix}$$

- ② N体系の波動関数を準粒子表示でのSlater行列式であるHFB状態として仮定する

$$|\Phi_{\text{HFB}}\rangle = \prod_i \alpha_i \alpha_i^\dagger |0\rangle$$

- ③ HFB状態を用いて粒子密度と対密度行列を求め、HFポテンシャルとペアポテンシャルを計算する。

$$\text{HF-} \quad \Gamma_{ij}[\rho] = \sum_{kl} \bar{v}_{ikjl} \rho_{lk} \quad \rho_{ij} = \langle \Phi_{\text{HFB}} | a_j^\dagger a_i | \Phi_{\text{HFB}} \rangle$$

$$\text{Pair-} \quad \Delta_{ij}[\kappa] = \frac{1}{2} \sum_{kl} \bar{v}_{ijkl} \kappa_{kl} \quad \kappa_{ij} = \langle \Phi_{\text{HFB}} | a_j a_i | \Phi_{\text{HFB}} \rangle$$

$$\sim \underbrace{V_0}_{\text{Pairing strength}} \left(1 - \underbrace{V_1}_{\text{Density dependence}} \left(\frac{\rho_q(\mathbf{r})}{\rho_0} \right)^\gamma \right) \kappa(\mathbf{r})$$

(Fitted to the OES data) $V_1=0$: volume
 $V_1=0.5$: mixed
 $V_1=1$: surface

※このペアポテンシャルはSkyrme相互作用に含まれておらず、複数のモデルがある。
 V_0 はペアリングの強さを表すパラメータ、
 V_1 はペアの核子の分布の仕方を表す。

④ HF- + pair- ハミルトニアンを対角化するように $w(u,v)$ を求める

$$\begin{pmatrix} e + \Gamma - \lambda & \Delta \\ -\Delta^* & -(e + \Gamma)^* + \lambda \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = E \begin{pmatrix} U \\ V \end{pmatrix}$$

⑤ ④で求めた $w(u,v)$ を用いて②に戻り④までくり返す。 $\delta E = 0$ になるまで②↔④を繰り返す（変分計算）。

$$\delta E[\rho, \kappa] = \frac{\langle \Phi_{\text{HFB}} | \hat{H} | \Phi_{\text{HFB}} \rangle}{\langle \Phi_{\text{HFB}} | \Phi_{\text{HFB}} \rangle} \quad \langle \Phi_{\text{HFB}} | \hat{N} | \Phi_{\text{HFB}} \rangle = n$$

拘束条件

Skyrme-Hartree Fock Bogoliubov 計算

↑
 ゼロレンジ力を用いた
 有効相互作用の種類

↑
 多体場を1体場に
 近似する理論

↑
 準粒子の導入

Skyrme相互作用

※Skyrme相互作用中のパラメータのセットには
多くの種類がある(2,300種類)

SLy シリーズ

SkM*

SAMi

SGII

etc.

$$H = \sum_i t_i + \sum_{i < j} v_{ij}^{(2)} + \sum_{i < j < k} v_{ijk}^{(3)}$$

1体 2体 3体

$$v_{12}^{(2)} = t_0(1 + x_0 \hat{P}_\sigma) \delta(\mathbf{r}_1 - \mathbf{r}_2) \leftarrow \text{ゼロレンジ力：同じ位置に存在する2粒子にのみ力が働く}$$

$\hat{P}_\sigma$: スピンを入れ替える演算子

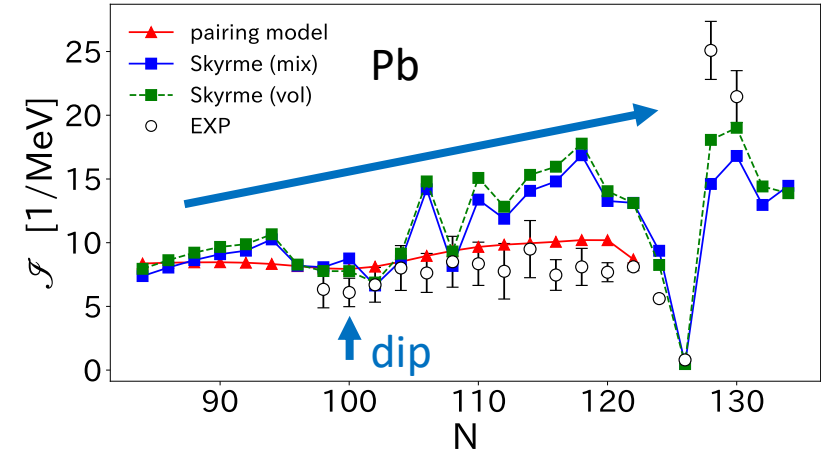
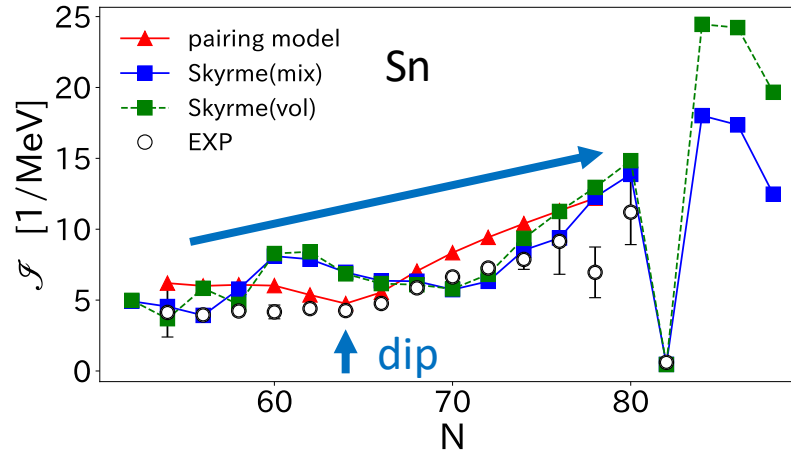
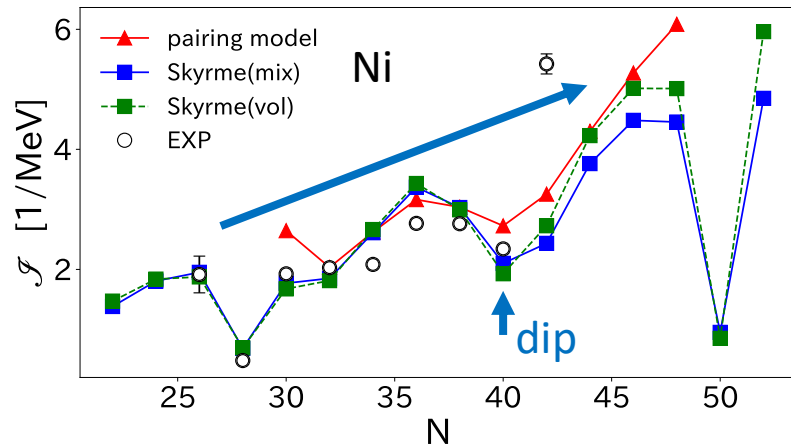
$$+ \frac{1}{2} t_1 (\delta(\mathbf{r}_1 - \mathbf{r}_2) \hat{k}^2 + \hat{k}'^2 \delta(\mathbf{r}_1 - \mathbf{r}_2))$$

$$+ t_2 \hat{k}^2 \cdot \delta(\mathbf{r}_1 - \mathbf{r}_2) \hat{k} + iW_0(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \hat{k}' \times \delta(\mathbf{r}_1 - \mathbf{r}_2) \hat{k}$$

$$\times \quad \hat{k} = \frac{1}{2i}(\nabla_1 - \nabla_2) , \quad \hat{k}' = -\frac{1}{2i}(\nabla_1 + \nabla_2)$$

対回転慣性モーメントの中性子数依存性

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※ペアリングの強度は ^{116}Sn , ^{68}Ni , ^{188}Pb のペアリングギャップでfit

$$J(N) = \frac{1}{\frac{\partial^2 E}{\partial N^2}} = \frac{4}{E_0(N-2) - 2E_0(N) + E_0(N+2)}$$

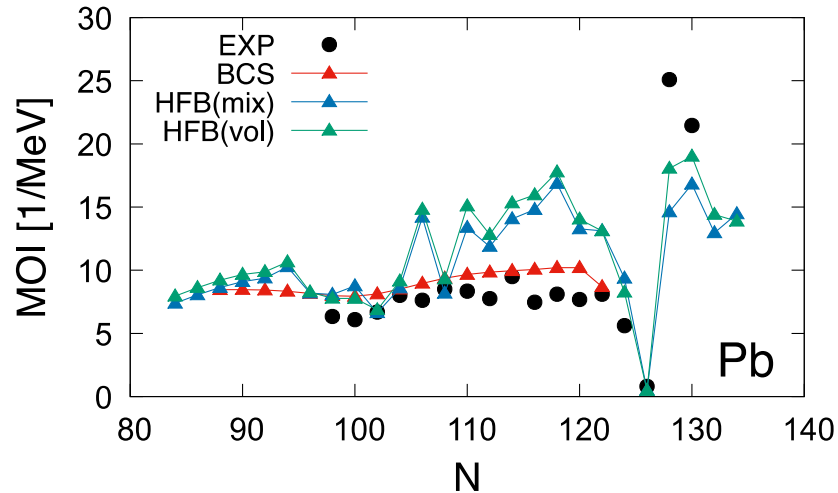
※ Skyrme Hartree-Fock-Bogoliubov : ゼロレンジカのSkyrme相互作用を用いた密度汎関数法

- ✓ 使用したコード : HFBTHO v2.00d M. V. Stoitsov et al.
Computer Physics Communications 184 (2013) 1592–1604
- ✓ 相互作用のパラメータセット : SLy4
- ✓ ペアリングモデル : mixed type, volume type

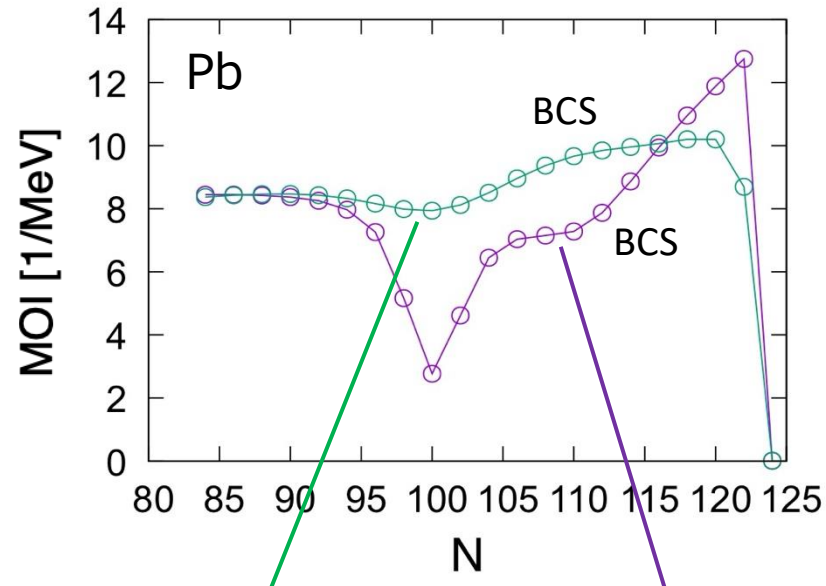
- 対回転慣性モーメントは中性子数に対して大まかに増加する
- 対回転慣性モーメントは同位体中央において値が落ち込む

N-MOI affected by the intruder orbit

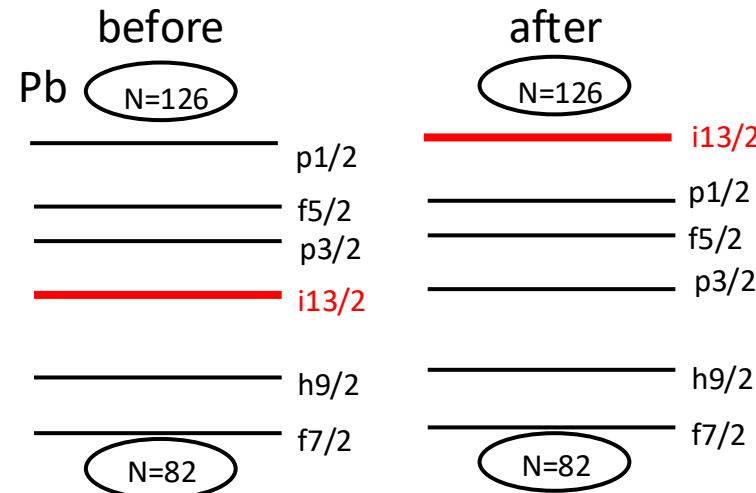
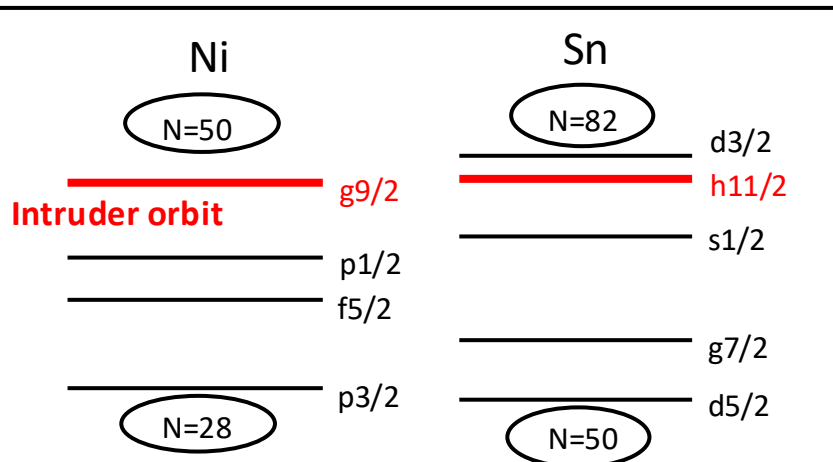
Why is the slope of N-MOI in Pb shallower than those in Ni and Sn?



$$MOI(N) = \frac{1}{\frac{\partial^2 H}{\partial N^2}} = \frac{4}{E_0(N-2) - 2E_0(N) + E_0(N+2)}$$



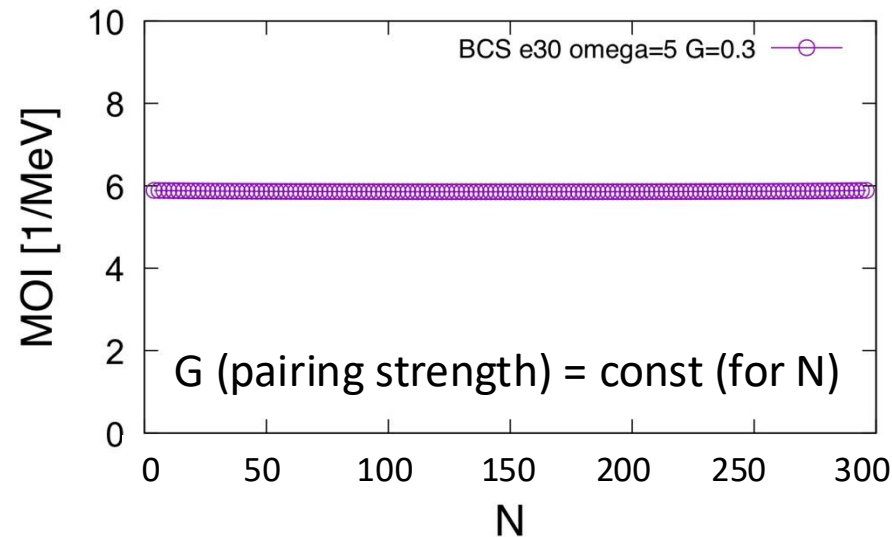
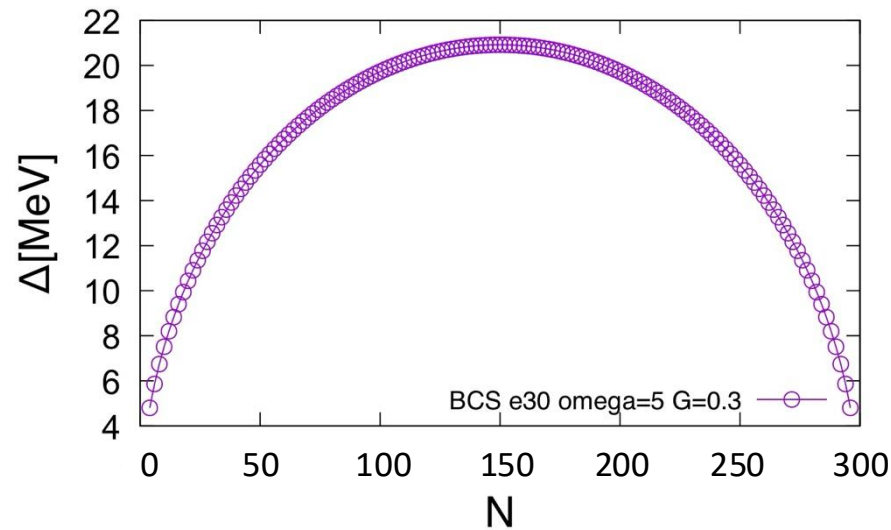
The trend of N-MOI is affected by the energy of an intruder orbit
 $\leftrightarrow$ The experimental trend of N-MOI tells us the energy ordering of the intruder orbit



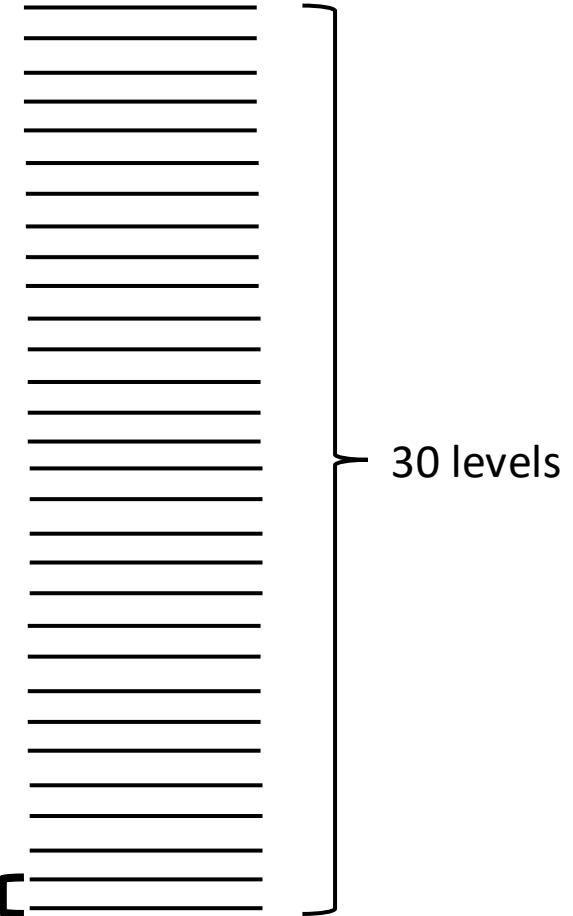
Pairing model with simplified single-particle levels

Why does MOI increase with N and change the trend of pairing MOI?

→ We adopt the schematic **30 single-particle levels**



$$MOI(N) = \frac{1}{\frac{\partial^2 H}{\partial N^2}} = \frac{4}{E_0(N-2) - 2E_0(N) + E_0(N+2)}$$



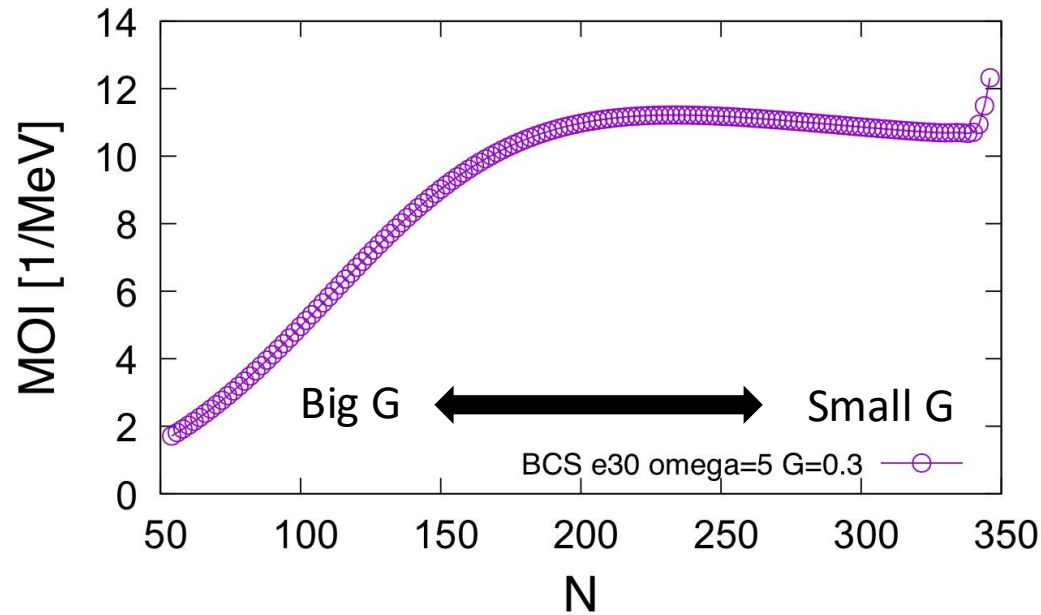
- Equal energy interval (1MeV)
- Equal degeneracy ($\Omega=5$)
- $G(\text{pairing strength})=\text{const}$ (for N)



MOI becomes constant for neutron number N

$\Omega = 5$
1MeV

Pairing model with simplified single-particle levels

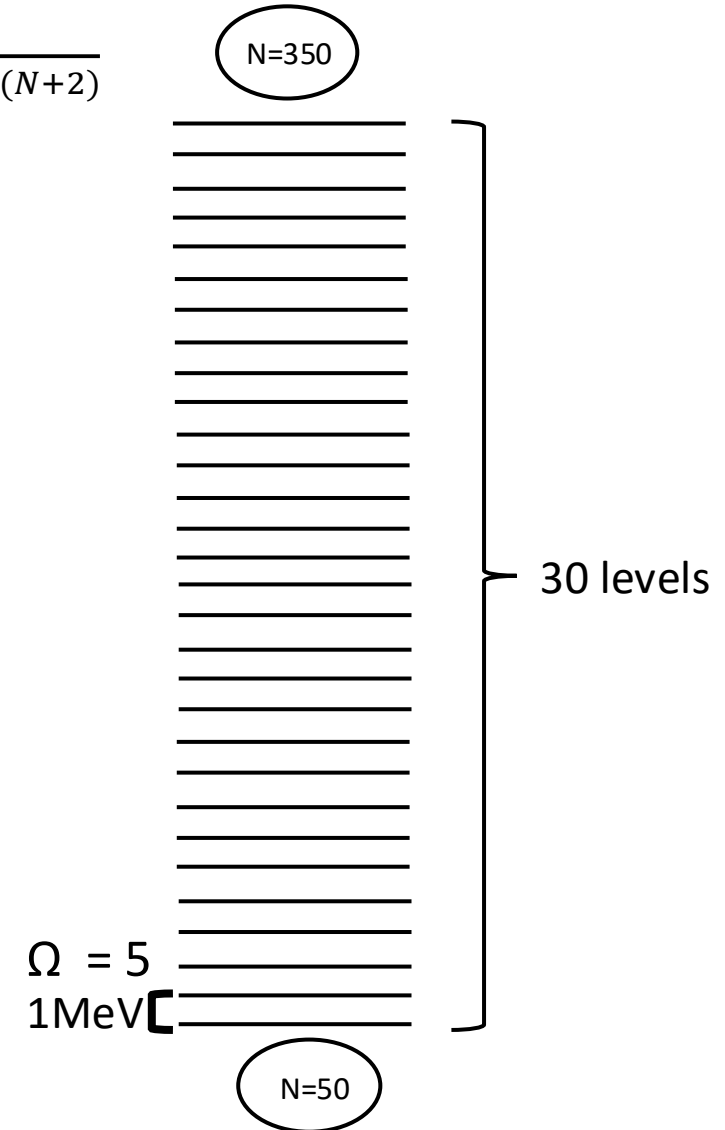


$$\text{MOI}(N) = \frac{1}{\frac{\partial^2 H}{\partial N^2}} = \frac{4}{E_0(N-2) - 2E_0(N) + E_0(N+2)}$$

- Equal space (1MeV)
- Equal degeneracy ($\Omega=5$)
- $G(\text{pairing strength}) = G(N_0) \times (N_0/N)$

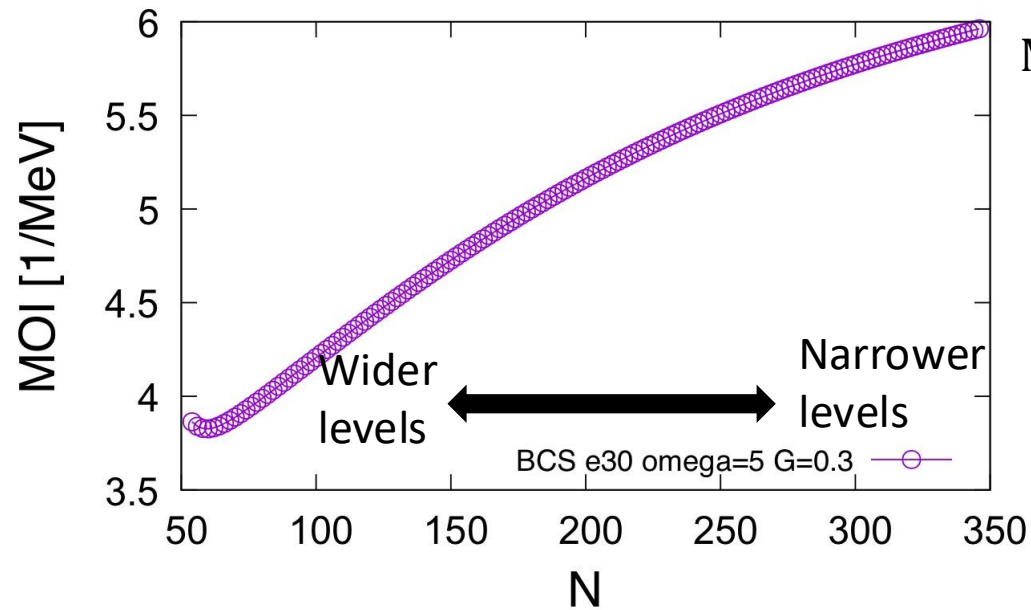


MOI **increases** with
neutron number N!

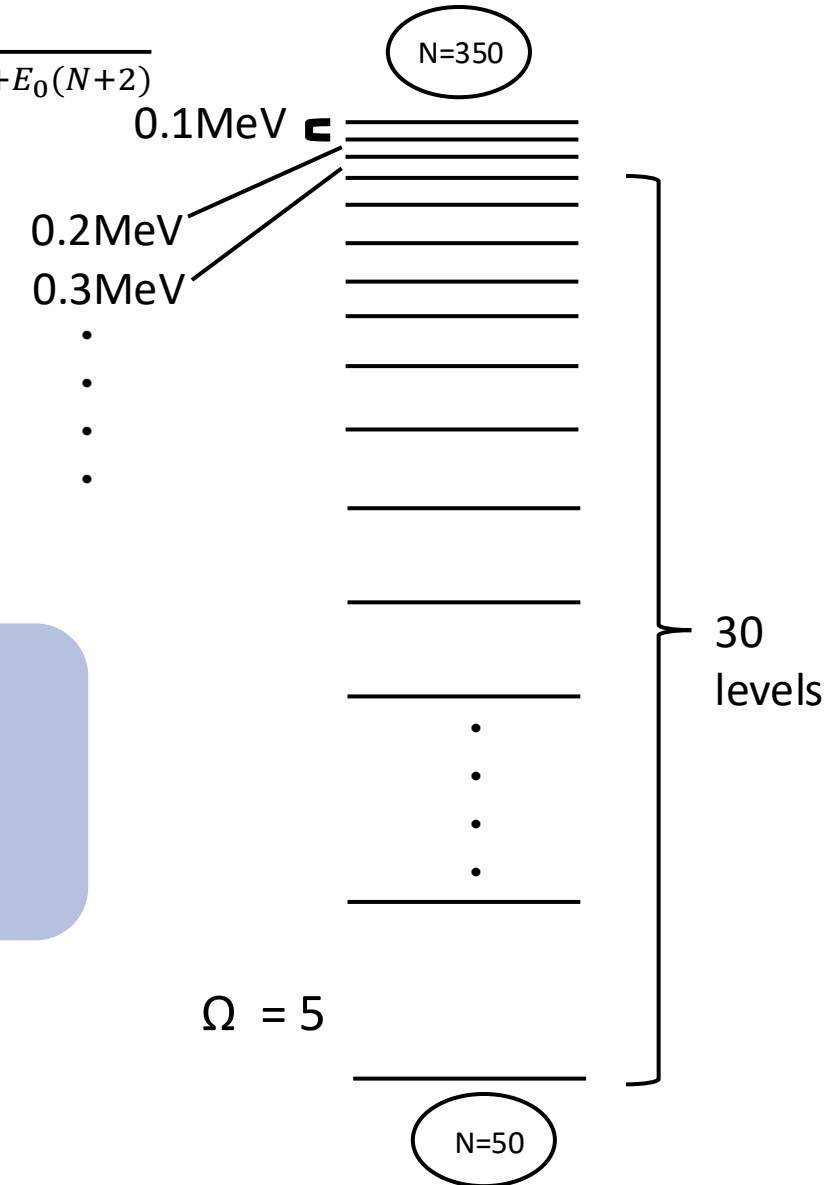


Pairing model with simplified single-particle levels

results 17



$$MOI(N) = \frac{1}{\frac{\partial^2 H}{\partial N^2}} = \frac{4}{E_0(N-2) - 2E_0(N) + E_0(N+2)}$$

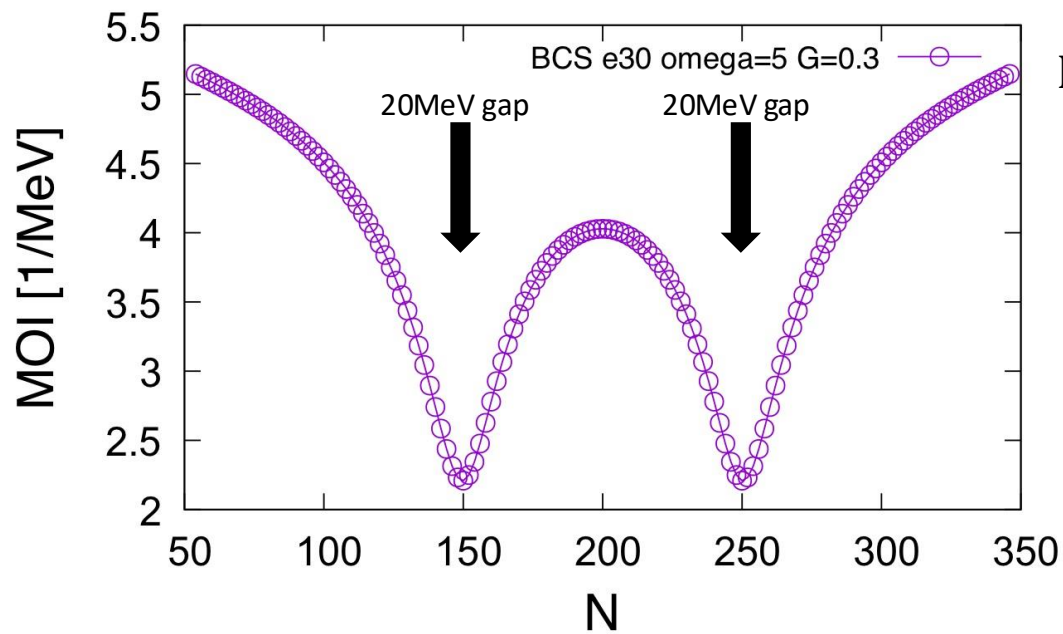


- Level density becomes bigger with N
- Equal degeneracy ($\Omega=5$)
- $G(\text{pairing strength})=\text{const}$ (for N)



MOI **increases** with neutron number N!

Pairing model with simplified single-particle levels



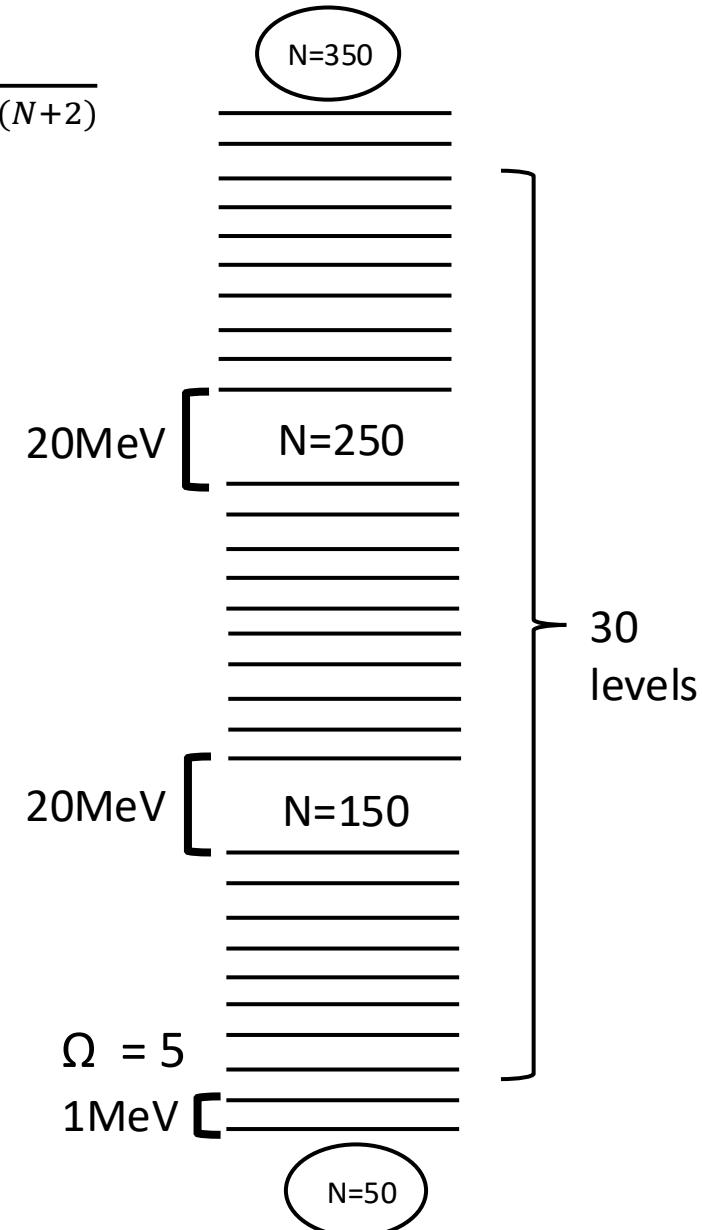
$$MOI(N) = \frac{1}{\frac{\partial^2 H}{\partial N^2}} = \frac{4}{E_0(N-2) - 2E_0(N) + E_0(N+2)}$$

- Equal space (1MeV)
- Equal degeneracy ($\Omega=5$)
- $G(\text{pairing strength}) = \text{const (for } N)$

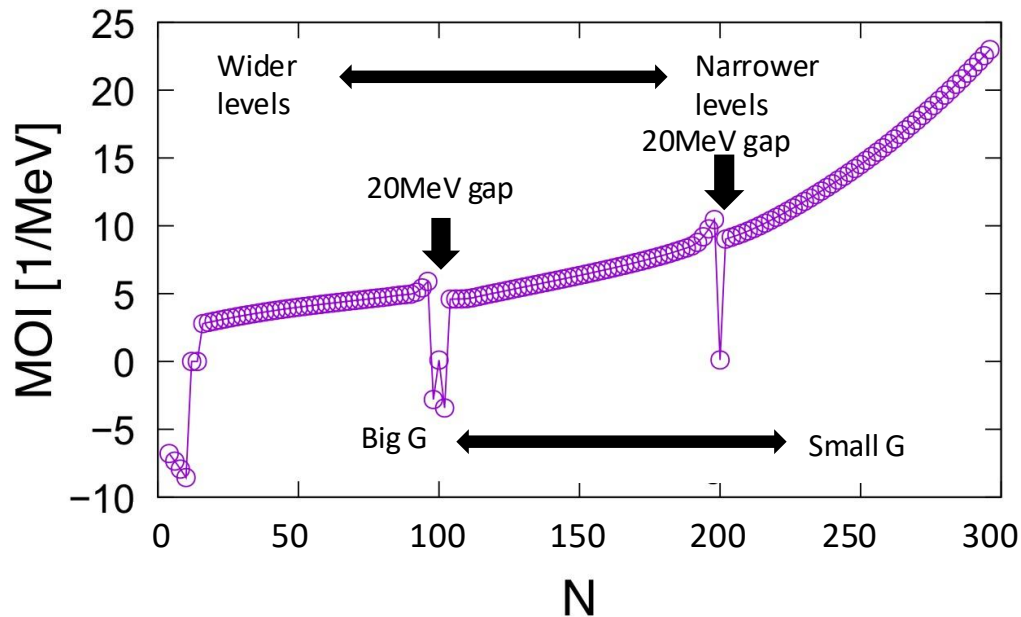
+ make shell gap above N=150&250



The **dip of MOI** appears
at N=150&250



Pairing model with simplified single-particle levels

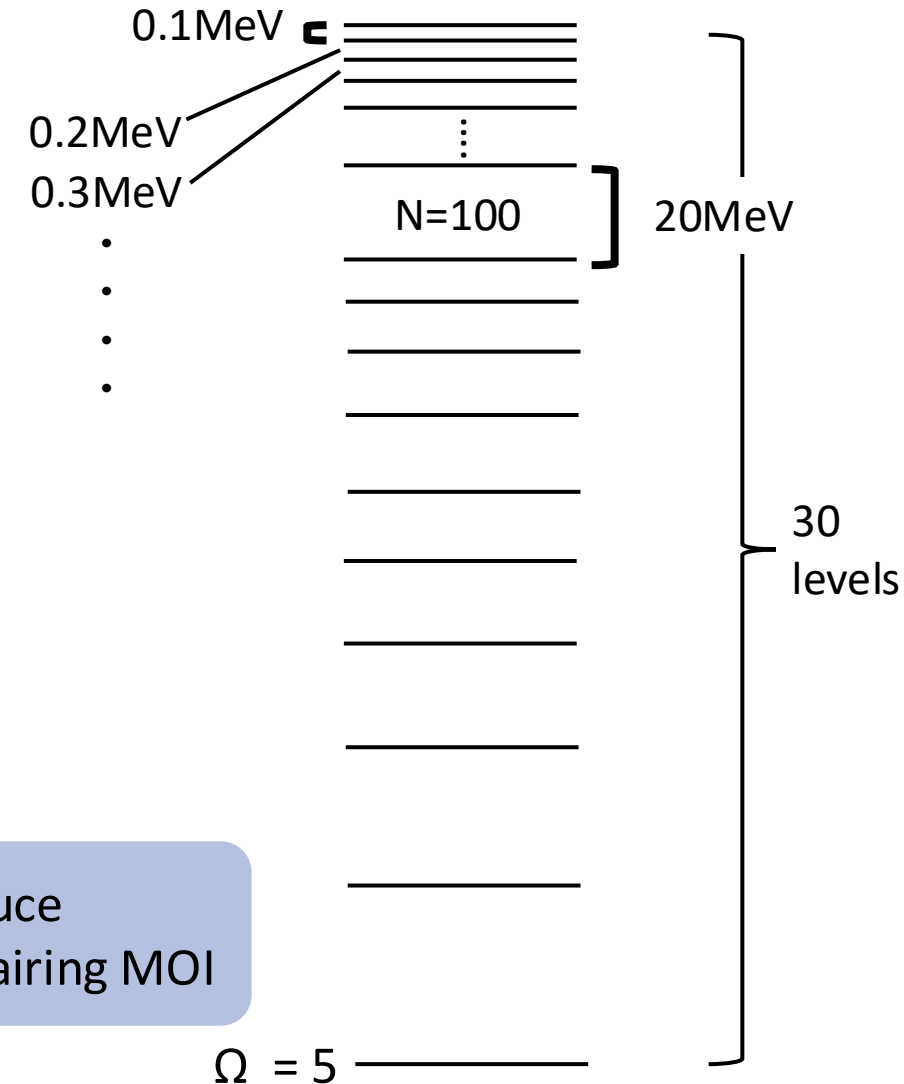


$$MOI(N) = \frac{1}{\frac{\partial^2 H}{\partial N^2}} = \frac{4}{E_0(N-2) - 2E_0(N) + E_0(N+2)}$$

- Level density becomes higher with N (position of the intruder orbit)
- Equal degeneracy ($\Omega=5$)
- $G(\text{pairing strength}) = G(N_0) \times (N_0/N)$
- + create shell gaps above N=100 & 200



Roughly reproduce the isotopic trend of pairing MOI

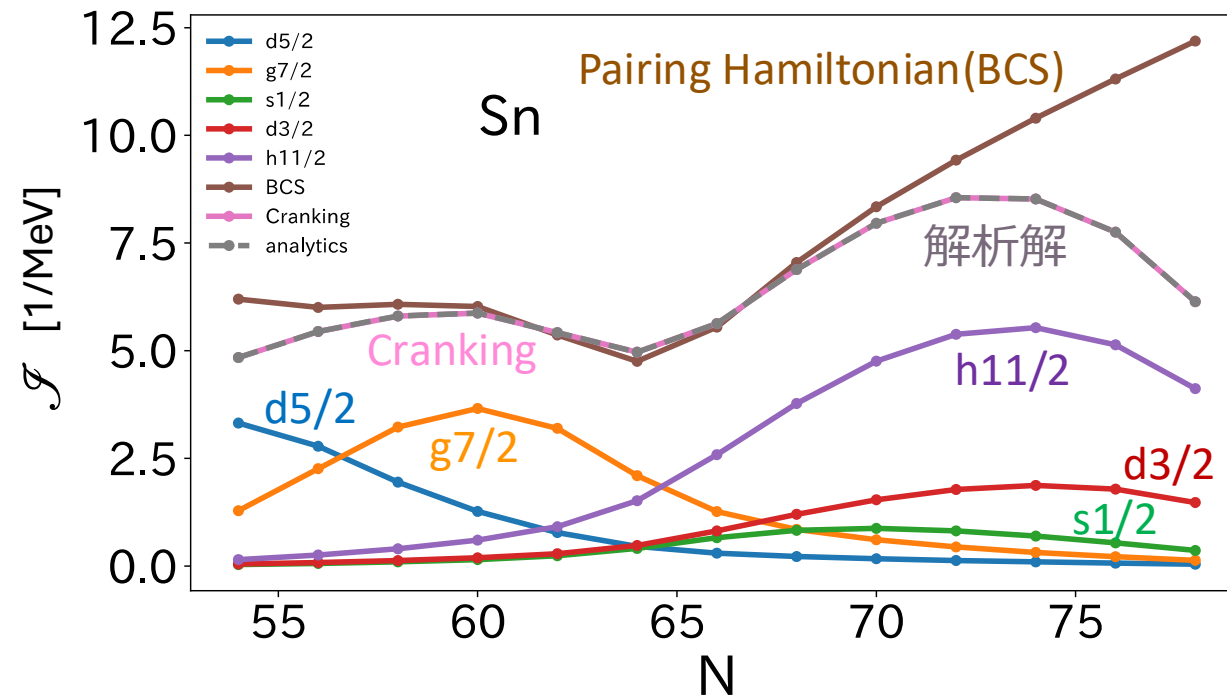


中性子数依存性における各軌道からの寄与

results 18

定性的な解析から、対回転慣性モーメントの各軌道からの寄与が計算できる

$$\mathcal{J} = \sum_{\nu} \frac{\Omega_{\nu}}{\Delta} \left[1 + \left(\frac{\epsilon_{\nu} - \lambda}{\Delta} \right)^2 \right]^{-3/2} \mathcal{J}_{\nu}$$



Pair amplitude dependence of pair MOI

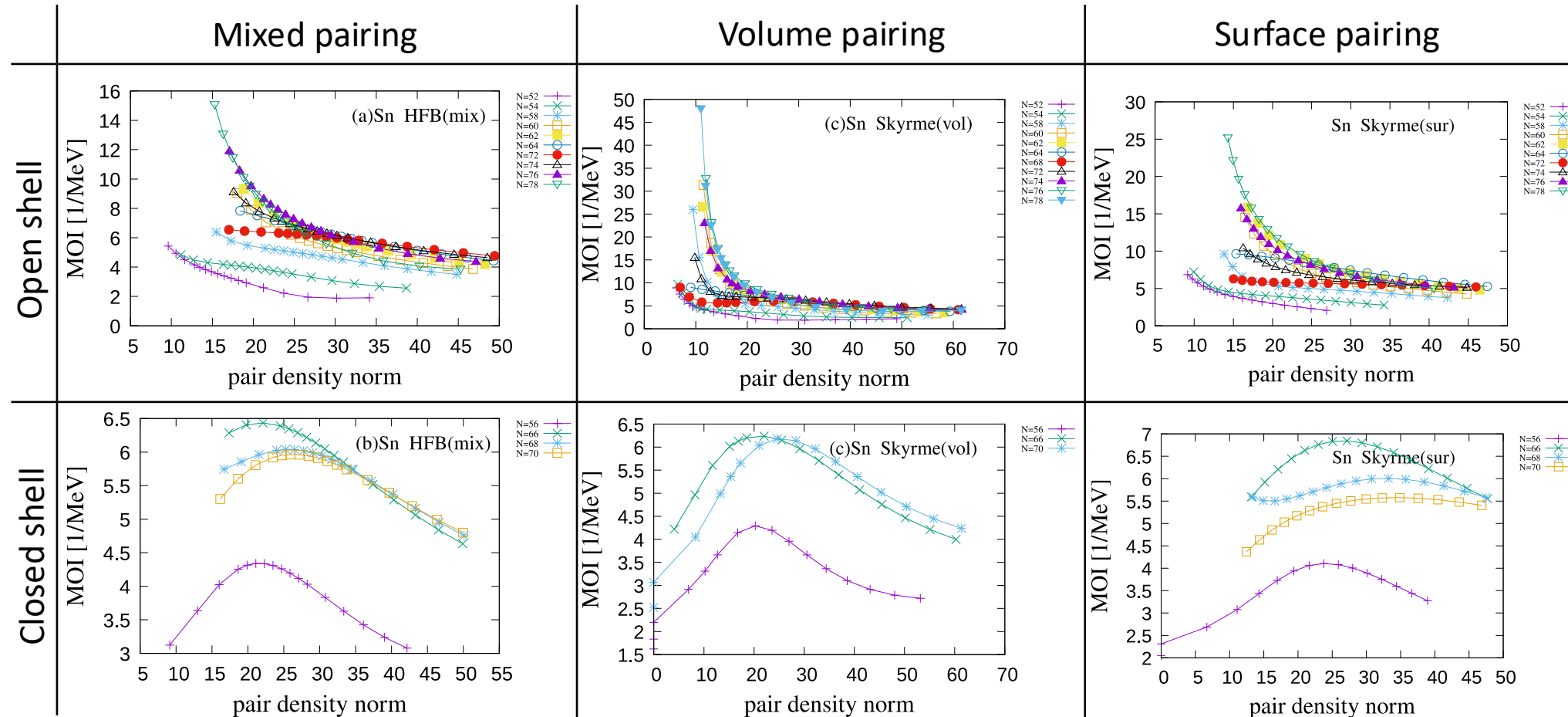
results 18

We calculated the pair amplitude dependence of the pair MOI with **Skyrme HFB (SLy4)** “pair density norm (order parameter)”

Pair density :

$$\tilde{\rho}(\mathbf{r}) = \sum_s (-2s) \langle c_{\mathbf{r}-s} c_{\mathbf{r}s} \rangle$$

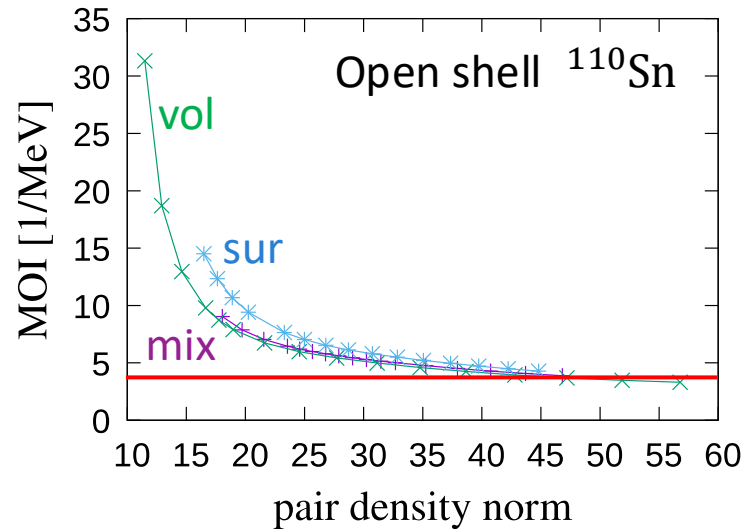
Integrated using the Gaussian quadratures
(same as integrating the particle density)



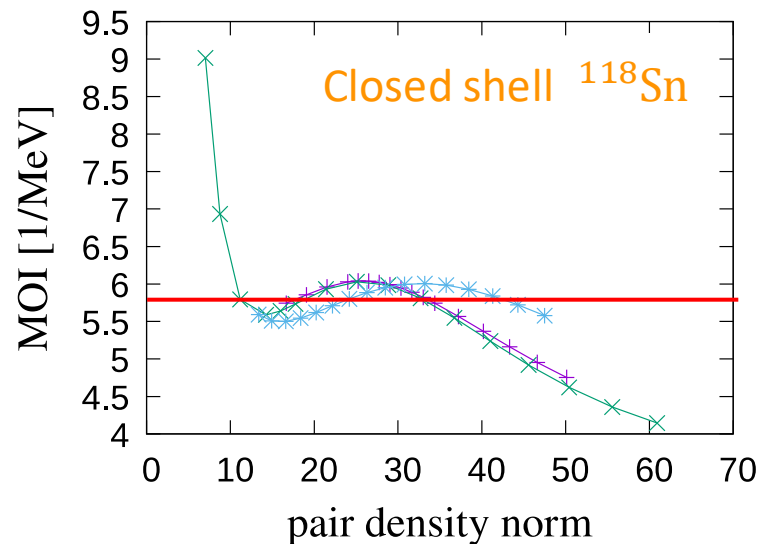
The results are almost the same as that of the pairing Hamiltonian, independent of the pairing type

Pair amplitude dependence of pair MOI

We calculated the pair amplitude dependence of the pair MOI with Skyrme HFB (SLy4)



- ✓ Less than 5 % differences between the pair amplitude of the pairing rotational moments of inertia obtained from experimental data using three types of pairing in the open-shell nuclei.



- ✓ In the closed-shell nuclei, for example, ^{118}Sn , there are two solutions with each pairing type
- ✓ In ^{120}Sn , there are no solutions with any type of pairing

“pair density norm (order parameter)”

Pair density :

$$\tilde{\rho}(\mathbf{r}) = \sum_s (-2s) \langle c_{\mathbf{r}-s} c_{\mathbf{r}s} \rangle$$

Integrated using the Gaussian quadratures (same as integrating the particle density)

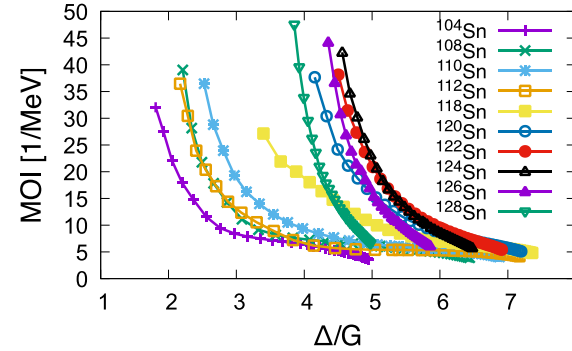
We should not fit the coefficient of the pairing term in nuclear EDF with closed-shell nuclei, especially with SLy4 interaction.

Qualitative proof of the pair amplitude dependence

discussion 9

We want to prove...

$$\frac{d\mathcal{J}}{d\Delta} < 0 \text{ in open-shell nuclei}$$



Analysis assuming the **continuous level density** in open-shell nuclei: $\rho(\epsilon) > 0$

$$\mathcal{J} = \frac{\partial N}{\partial \lambda} \rightarrow \frac{\partial \mathcal{J}}{\partial \Delta} = \frac{\partial^2 N}{\partial \lambda \partial \Delta}$$

$$\frac{\partial \mathcal{J}}{\partial \Delta} = \frac{\partial^2 N}{\partial \lambda \partial \Delta} = - \int_{-\Lambda}^{\Lambda} 2\rho(\tilde{\epsilon} + \lambda) \frac{\partial^2 g(\tilde{\epsilon}, \Delta)}{\partial \tilde{\epsilon} \partial \Delta} d\tilde{\epsilon}$$

$$= -\frac{1}{2\Delta} \int_{-\Lambda/\Delta}^{\Lambda/\Delta} 2\rho(\lambda + x\Delta) \frac{1 - 2x^2}{(x^2 + 1)^{5/2}} dx.$$

$$= -\frac{2\rho(\lambda)}{\Delta} \left[\int_0^{\eta} dx \frac{1 - 2x^2}{(x^2 + 1)^{5/2}} \right] < 0$$

$$\equiv I(\eta) > 0 \quad (\text{always positive})$$

Particle number

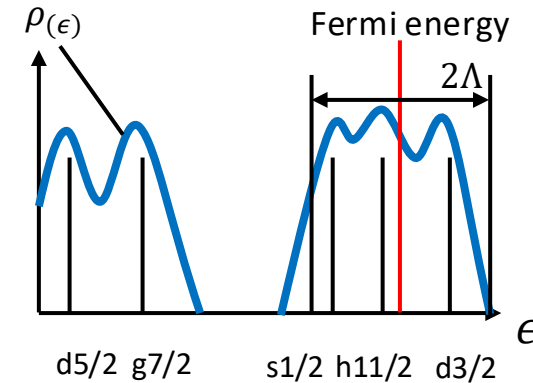
$$N(\lambda, \Delta) = \int_{\lambda-\Lambda}^{\lambda+\Lambda} 2\rho(\epsilon) g(\epsilon - \lambda, \Delta) d\epsilon$$

$$g(\tilde{\epsilon}, \Delta) \equiv \frac{1}{2} \left(1 - \frac{\tilde{\epsilon}}{\sqrt{\tilde{\epsilon}^2 + \Delta^2}} \right)$$

Occupation probability (BCS approx.)

$$\rho(\lambda + x\Delta) \approx \rho(\lambda) + x\Delta\rho'(\lambda)$$

approx.

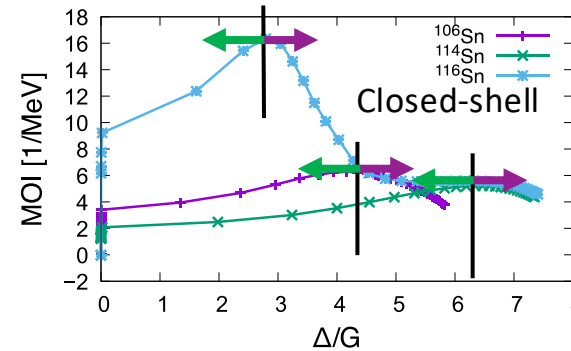


Qualitative proof of the pair amplitude dependence

discussion 10

We want to prove...

$\frac{d\mathcal{J}}{d\Delta} > 0$ in closed-shell nuclei (for small Δ)



$\frac{d\mathcal{J}}{d\Delta} > 0$, $\frac{d\mathcal{J}}{d\Delta} < 0$
Same as open-shell nuclei

closed-shell nuclei: $\rho(\epsilon) = 0$ consider $\tilde{\rho}(\epsilon) > 0$ even around the fermi energy

$$\mathcal{J} = \frac{\partial N}{\partial \lambda} = \frac{\partial}{\partial \lambda} \left[\int_{\lambda-\Lambda}^{\lambda+\Lambda} 2\tilde{\rho}(\epsilon)g(\epsilon-\lambda, \Delta)d\epsilon - \int_{\lambda-\frac{\delta\epsilon}{2}}^{\lambda+\frac{\delta\epsilon}{2}} 2\tilde{\rho}(\epsilon)g(\epsilon-\lambda, \Delta)d\epsilon \right]$$

Occupation probability (BCS approx.)

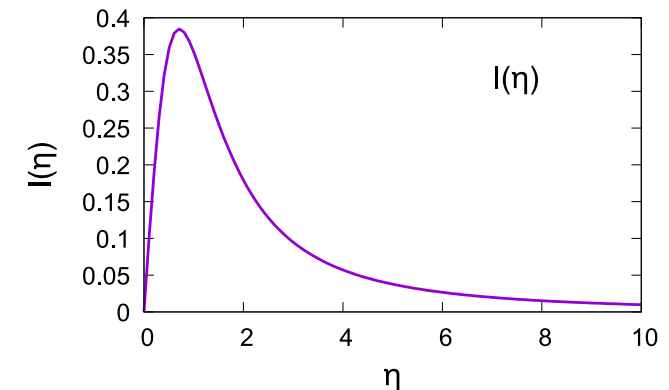
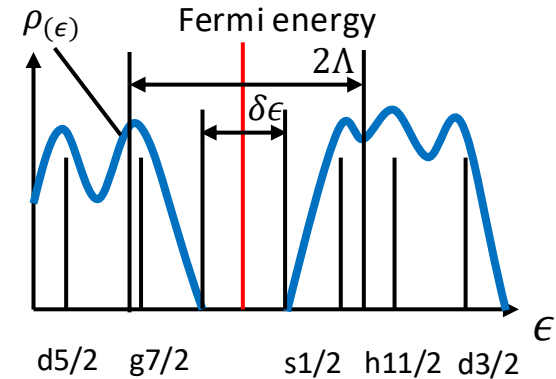
$$g(\tilde{\epsilon}, \Delta) \equiv \frac{1}{2} \left(1 - \frac{\tilde{\epsilon}}{\sqrt{\tilde{\epsilon}^2 + \Delta^2}} \right)$$

$$\frac{\partial \mathcal{J}}{\partial \Delta} = -\frac{1}{2\Delta} \left[\int_{-\Lambda/\Delta}^{\Lambda/\Delta} 2\tilde{\rho}(\lambda+x\Delta) \frac{1-2x^2}{(x^2+1)^{5/2}} dx - \int_{-\delta\epsilon/2\Delta}^{\delta\epsilon/2\Delta} 2\tilde{\rho}(\lambda+x\Delta) \frac{1-2x^2}{(x^2+1)^{5/2}} dx \right]$$

approx. $\tilde{\rho}(\lambda+x\Delta) \approx \tilde{\rho}(\lambda) + x\Delta\tilde{\rho}'(\lambda)$

$$= -\frac{2\tilde{\rho}(\lambda)}{\Delta} \left[I\left(\frac{\Lambda}{\Delta}\right) - I\left(\frac{\delta\epsilon}{2\Delta}\right) \right] > 0$$

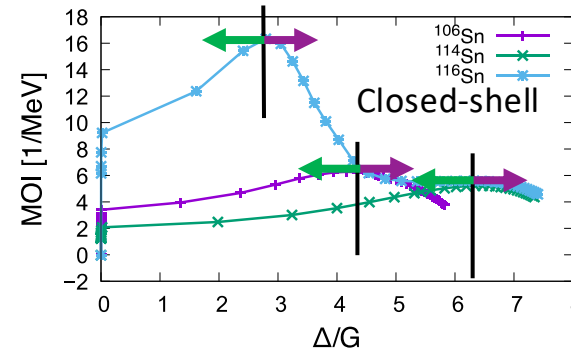
$< 0 \quad (\Lambda \gg \delta\epsilon)$



Qualitative proof of the pair amplitude dependence

We want to prove...

$\frac{d\mathcal{J}}{d\Delta} > 0$ in closed-shell nuclei (for small Δ)



$\frac{d\mathcal{J}}{d\Delta} > 0$, $\frac{d\mathcal{J}}{d\Delta} < 0$
Same as open-shell nuclei

closed-shell nuclei: $\rho(\epsilon) = 0$ consider $\tilde{\rho}(\epsilon) > 0$ even around the fermi energy

$$\mathcal{J} = \frac{\partial N}{\partial \lambda} = \frac{\partial}{\partial \lambda} \left[\int_{\lambda-\Lambda}^{\lambda+\Lambda} 2\tilde{\rho}(\epsilon)g(\epsilon-\lambda, \Delta)d\epsilon - \int_{\lambda-\frac{\delta\epsilon}{2}}^{\lambda+\frac{\delta\epsilon}{2}} 2\tilde{\rho}(\epsilon)g(\epsilon-\lambda, \Delta)d\epsilon \right]$$

Occupation probability (BCS approx.)

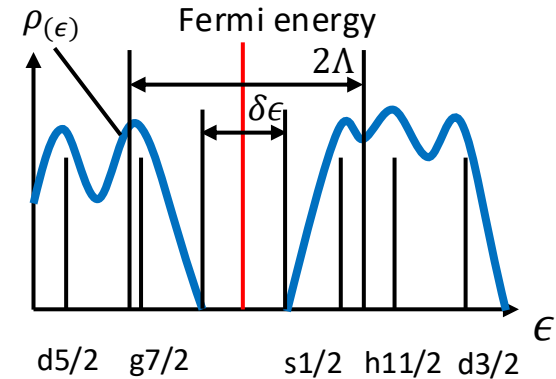
$$g(\tilde{\epsilon}, \Delta) \equiv \frac{1}{2} \left(1 - \frac{\tilde{\epsilon}}{\sqrt{\tilde{\epsilon}^2 + \Delta^2}} \right)$$

$$\frac{\partial \mathcal{J}}{\partial \Delta} = -\frac{1}{2\Delta} \left[\int_{-\Lambda/\Delta}^{\Lambda/\Delta} 2\tilde{\rho}(\lambda+x\Delta) \frac{1-2x^2}{(x^2+1)^{5/2}} dx - \int_{-\delta\epsilon/2\Delta}^{\delta\epsilon/2\Delta} 2\tilde{\rho}(\lambda+x\Delta) \frac{1-2x^2}{(x^2+1)^{5/2}} dx \right]$$

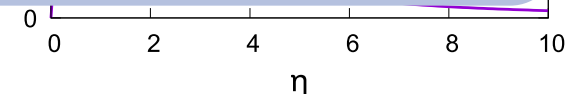
$$= -\frac{2\tilde{\rho}(\lambda)}{\Delta} \left[I\left(\frac{\Lambda}{\Delta}\right) - I\left(\frac{\delta\epsilon}{2\Delta}\right) \right] > 0$$

< 0 ($\Lambda \gg \delta\epsilon$)

approx.



Qualitative proof of the order parameter dependence of the pairing MOI can be constructed using the BCS approximation in both open-shell and closed-shell



対凝縮量依存性に関する定性的な解析

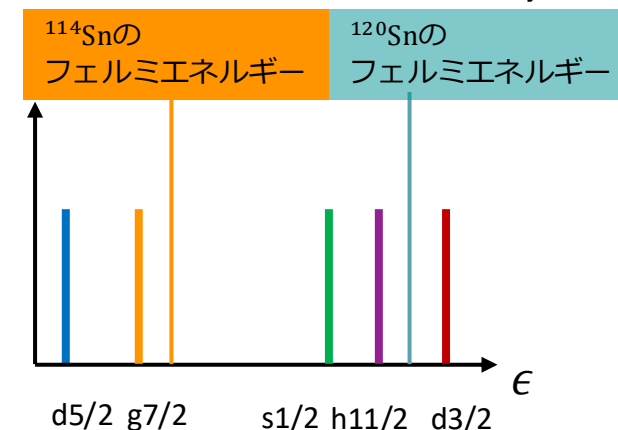
discussion 12

対回転慣性モーメントの各軌道からの寄与を分割して計算

$$\mathcal{J} = \int_{-\infty}^{\infty} \rho(\tilde{\epsilon} + \lambda) \frac{\Delta^2}{(\tilde{\epsilon}^2 + \Delta^2)^{3/2}} d\tilde{\epsilon} \quad - \star$$

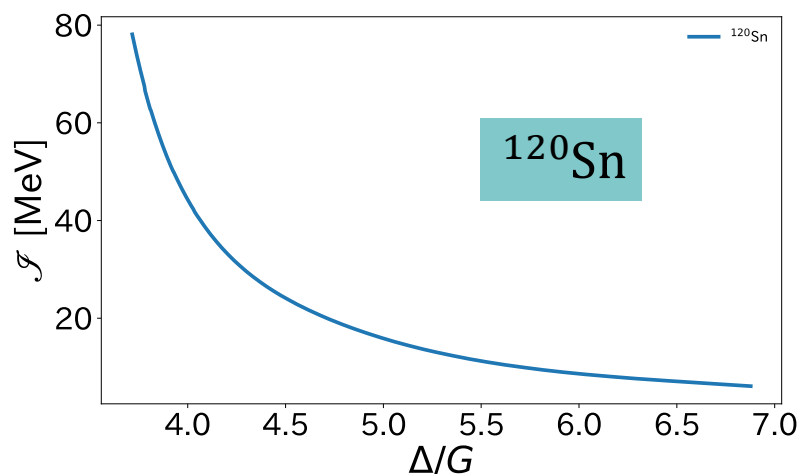
準位密度を離散的なものに置き換える

$$\mathcal{J} = \sum_{\nu} \frac{\Omega_{\nu}}{\Delta} \left[1 + \left(\frac{\epsilon_{\nu} - \lambda}{\Delta} \right)^2 \right]^{-3/2} \mathcal{J}_{\nu}$$

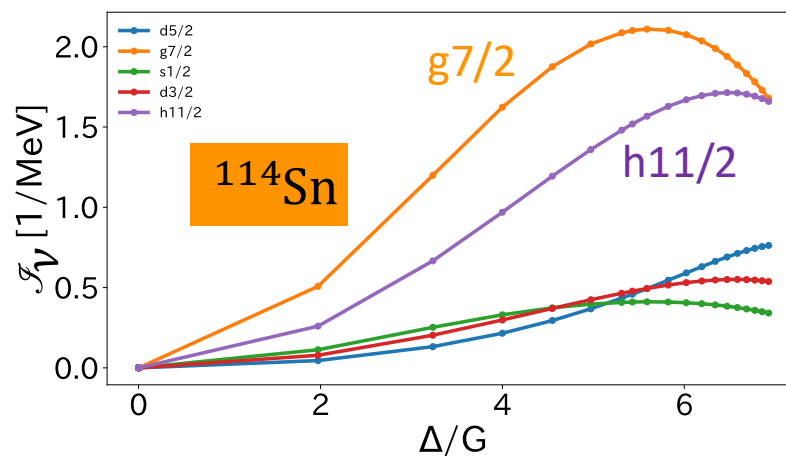
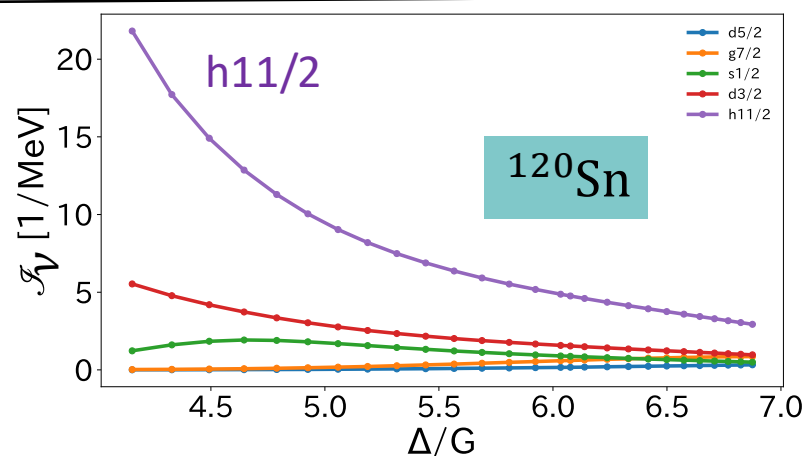
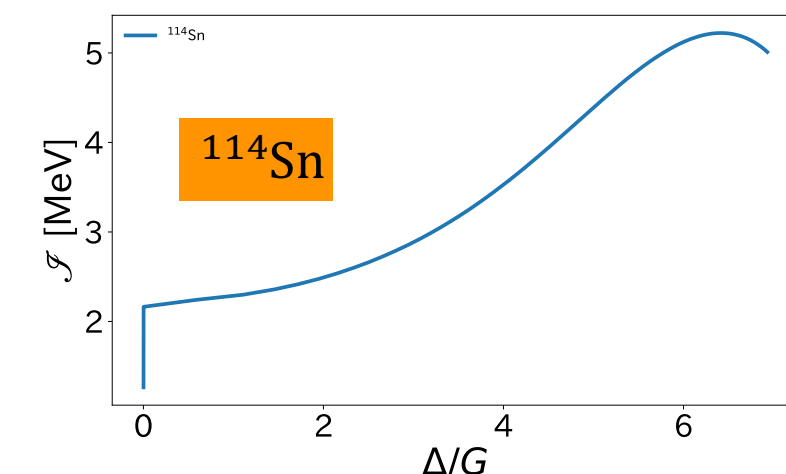


フェルミエネルギーに最も近い1粒子準位が対回転慣性モーメントに最大の寄与をしている

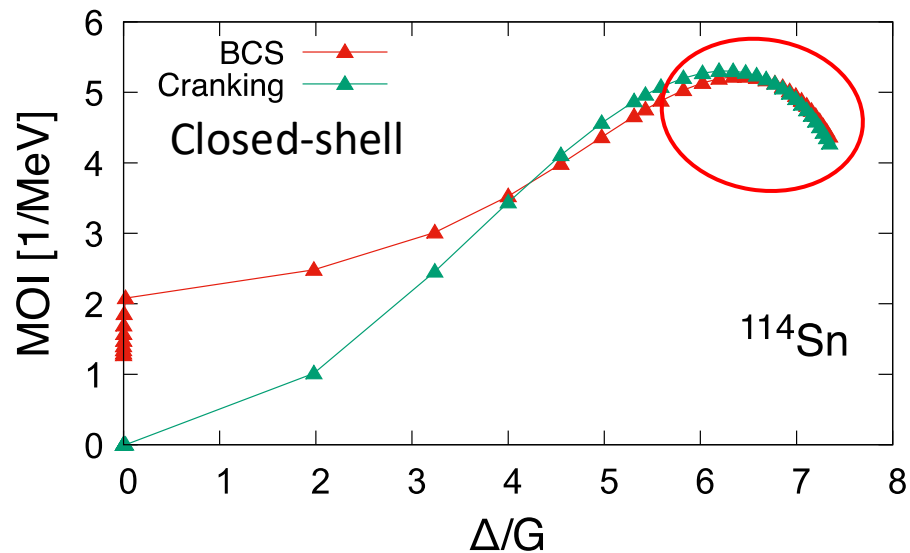
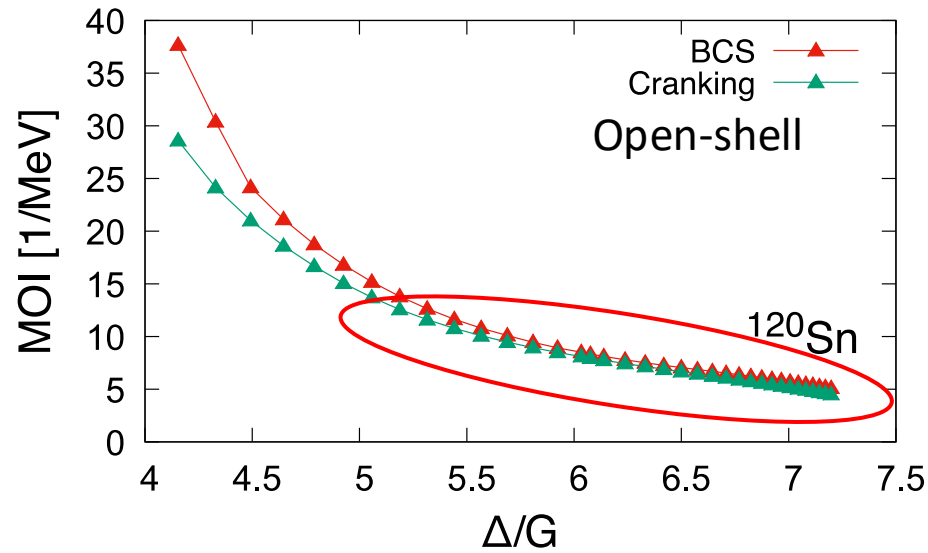
Open shell



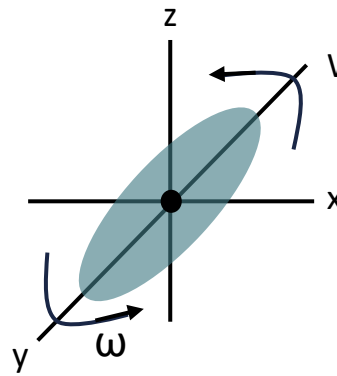
Closed shell



- Qualitative proof of the order parameter dependence using the **cranking** formula



Cranking MOI: Assuming the axisymmetrically deformed nucleus as the rigid body, we rotate and get the inertial mass around the symmetry axis.



$$(H - \hbar\omega J_x) |\Phi\rangle = E' |\Phi\rangle$$

We get the angular momentum as a perturbative expansion

$$J_{\text{crank}} = 2 \sum_{l>0} \frac{|\langle \nu\bar{\nu} | \hat{N} | \text{BCS} \rangle|^2}{E_l + E_{\bar{l}}} = \sum_{l>0} \frac{4\Omega_l U_l^2 V_l^2}{E_l}. \quad (1)$$

Open-shell

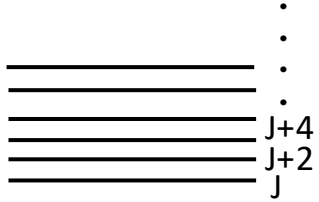
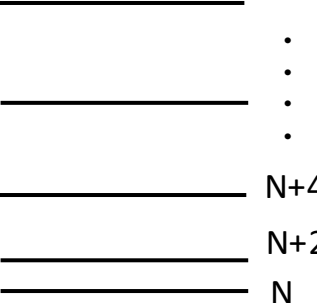
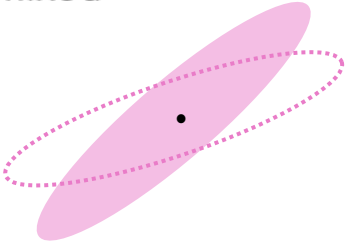
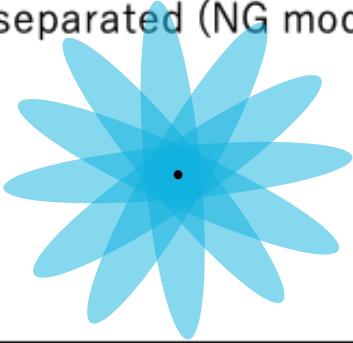
$$J_{\text{crank}} = \int_{\lambda-\Lambda}^{\lambda+\Lambda} \frac{4\rho(\epsilon)}{\sqrt{(e_\nu - \lambda)^2 + \Delta^2}} \frac{g(\epsilon - \lambda, \Delta)[1 - g(\epsilon - \lambda, \Delta)] d\epsilon}{U^2 V^2} \quad (2)$$

$$= 2\rho(\lambda) \frac{\Lambda/\Delta}{\sqrt{\Lambda^2/\Delta^2 + 1}}$$

$$\frac{\partial J_{\text{crank}}}{\partial \Delta} = -\frac{2\rho(\lambda)}{\Delta} \frac{\Lambda/\Delta}{(\Lambda^2/\Delta^2 + 1)^{3/2}} = -\frac{2\rho(\lambda)}{\Delta} I(\Lambda/\Delta) < 0 \quad (3)$$

The continuous limit of cranking pair-MOI matches that of BCS pair-MOI, especially in the range of large pair-condensation (small shell effect).

■ The order parameter dependence of the pairing MOI is different from that of the spatial one.

	spatial rotation	pairing rotation
Order parameter	$\beta \rightarrow \text{larger}$	$\Delta / G \rightarrow \text{larger}$
Moment of inertia	larger	smaller
Spacing of band	narrower 	wider 
intrinsic state	J mixed 	N separated (NG mode) 

Spatial MOI

Difficult to rotate

Pairing MOI

Easy to rotate??

$$\frac{1}{A\langle r^2 \rangle} \langle \Phi | \hat{Q}_{20} | \Phi \rangle \propto \beta$$

β is defined by the **ph** operator

$$\hat{Q}_{20} = \sum_{i,j} \langle i | r Y_{20} | j \rangle \boxed{c_i^\dagger c_j}$$

$$\langle \Phi | \sum_i c_i^\dagger c_i | \Phi \rangle \sim \Delta / G$$

Δ / G is defined by the **pp(hh)** operator

$$\sum_i c_i^\dagger c_i = \sum_{i,j} \langle i | \hat{T} | j \rangle \boxed{c_i c_j}$$

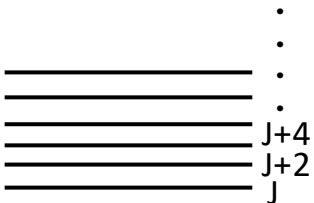
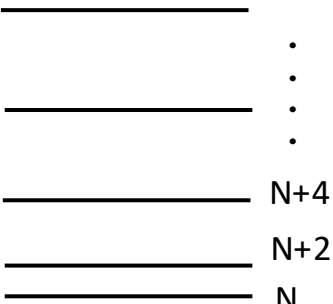
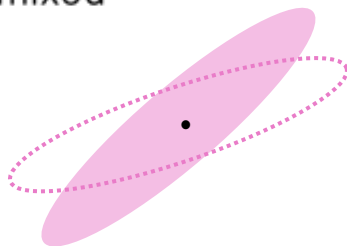
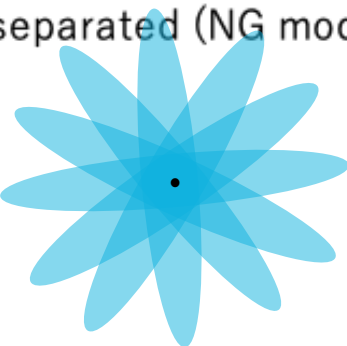
spatial

$$H' = V - \chi Q^\dagger Q - G P^\dagger P - \omega J_x \quad G \sim 0$$

Pairing

$$H' = V - \chi Q^\dagger Q - G P^\dagger P - \lambda N \quad x \sim 0$$

The reason why the property of the MOI differs between spatial rotation and pair rotation is the **difference in the operators** that make up the interaction and order parameters.

	spatial rotation	pairing rotation
Order parameter	$\beta \rightarrow$ larger	$\Delta / G \rightarrow$ larger
Moment of inertia	larger	smaller
Spacing of band	narrower 	wider 
intrinsic state $ \Phi_{\text{def}}\rangle = \sum_J C_J J\rangle$ $H J\rangle = E J\rangle$ $E_0 = E_1 = E_2 = \dots = E_{J=n} = E$ $H \Phi_{\text{def}}\rangle = \sum_J C_J H J\rangle = \sum_J C_J E_J J\rangle$ $= nE \sum_J C_J J\rangle = nE \Phi_{\text{def}}\rangle$	J mixed 	N separated (NG mode) 

There are two types of “stable”

- ✓ Small quantum fluctuation
- ✓ Small ground-state energy

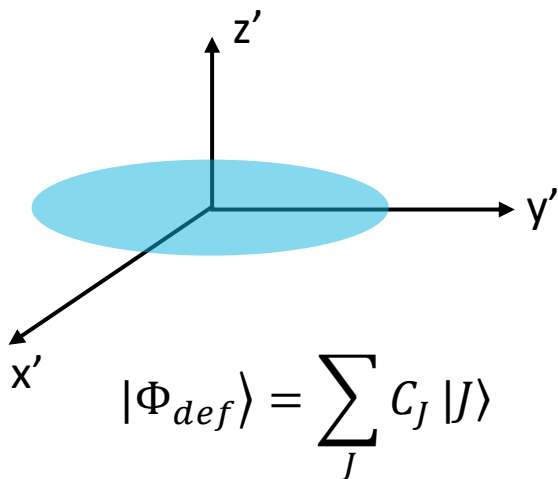
The state that has smaller energy selected

??

That’s why the pair rotational mode appears even it has a large quantum fluctuation

物体固定系（対称性が破れている）

空間回転



角運動量射影

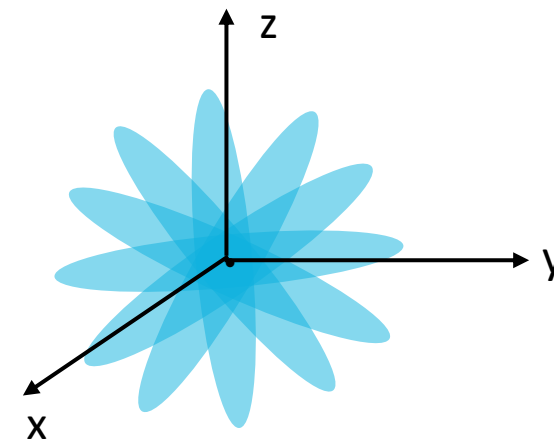
様々な角運動量 J を含む物体固定系での変形状態 $|\Phi_{def}\rangle$ に角運動量射影演算子（ある角運動量の状態のみを取り出し位相角 θ で積分する）を作用させる

→状態は特定の方角を持たなくなり、実験室系では球形となる。その一方、特定の J のみを含む状態を取り出せる

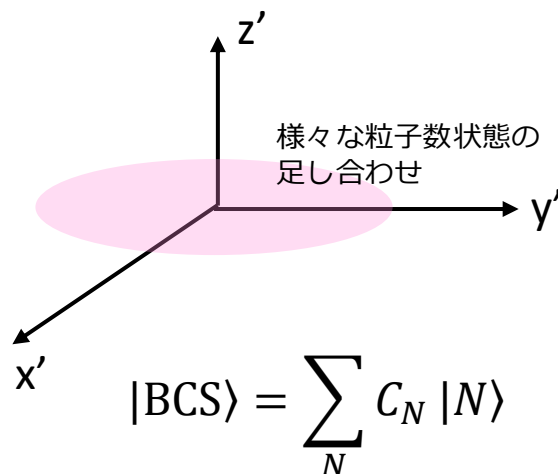
→回転バンドが定義できる

$$P_J |\Phi_{def}\rangle = \frac{1}{2\pi} \int d\theta e^{i(\hat{J}-J)\theta} |\Phi_{def}\rangle = |J\rangle$$

実験室系（対称性回復）



対回転



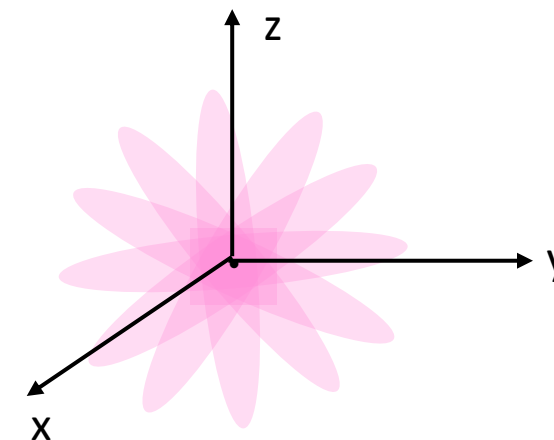
粒子数射影

様々な粒子数 N を含む物体固定系での変形状態 $|\text{BCS}\rangle$ に粒子数射影演算子（ある粒子数の状態のみを取り出しゲージ角 ϕ で積分する）を作用させる

→状態は特定の方角を持たなくなり、実験室系では球形となる。その一方、特定の N のみを含む状態を取り出せる

→対回転バンドが定義できる

$$P_N |\text{BCS}\rangle = \frac{1}{2\pi} \int d\phi e^{i(\hat{N}-N)\phi} |\text{BCS}\rangle = |N\rangle$$



$$\delta \langle \hat{H} - \omega \hat{J} \rangle = 0 \quad \begin{array}{l} \langle J \rangle \text{ is fixed,} \\ \text{and } \omega \& \beta \text{ are lead self-consistently} \end{array}$$

① To do this calculation, we use Nilsson code from Nakatsukasa-san

② In this Nilsson code, we solve this Hamiltonian

$$H = -\frac{\hbar^2}{2M}\Delta + \frac{M}{2}[\bar{\omega}_{\perp}^2(x^2 + y^2) + \bar{\omega}_z^2 z^2] - Cl \cdot s - D(l^2 - \langle l^2 \rangle) - \hbar\omega J_x$$

$$\bar{\omega}_z = \bar{\omega}(1 - \frac{2}{3}\beta)$$

$$\bar{\omega}_{\perp} = \bar{\omega}(1 + \frac{1}{3}\beta)$$

③ But we cannot fix $\langle J \rangle$, we set $\beta \& \omega$ to reproduce a specific value of $\langle J \rangle$ (=14)

④ we set various β and calculate MOI of spatial rotation

$$I^{-1} = \frac{\partial^2 E}{\partial^2 J} = \frac{\partial \omega}{\partial J} = \frac{\omega(J) - \omega(J-2)}{2} \quad (= \frac{\omega(14) - \omega(12)}{2})$$



MOI of spatial rotation Angular momentum

With fixed β

$$N_{sh} = 2 - 8$$

Orbitals have same parity can correlate each other

β dependence of MOI in spatial deformation

results 26

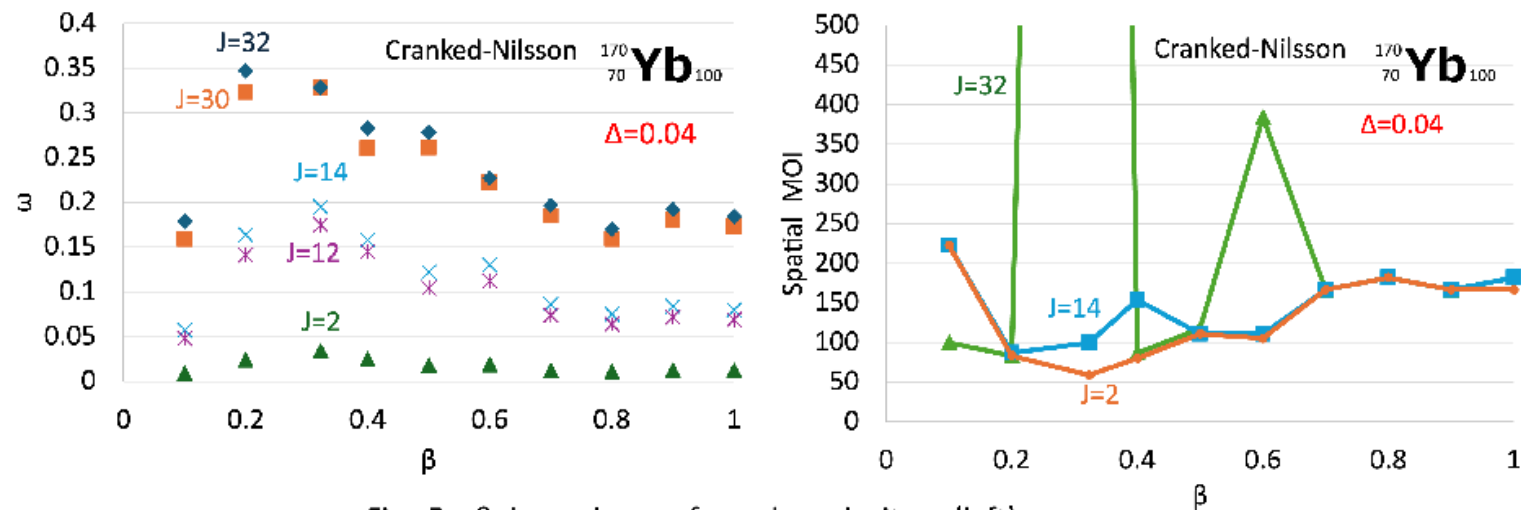
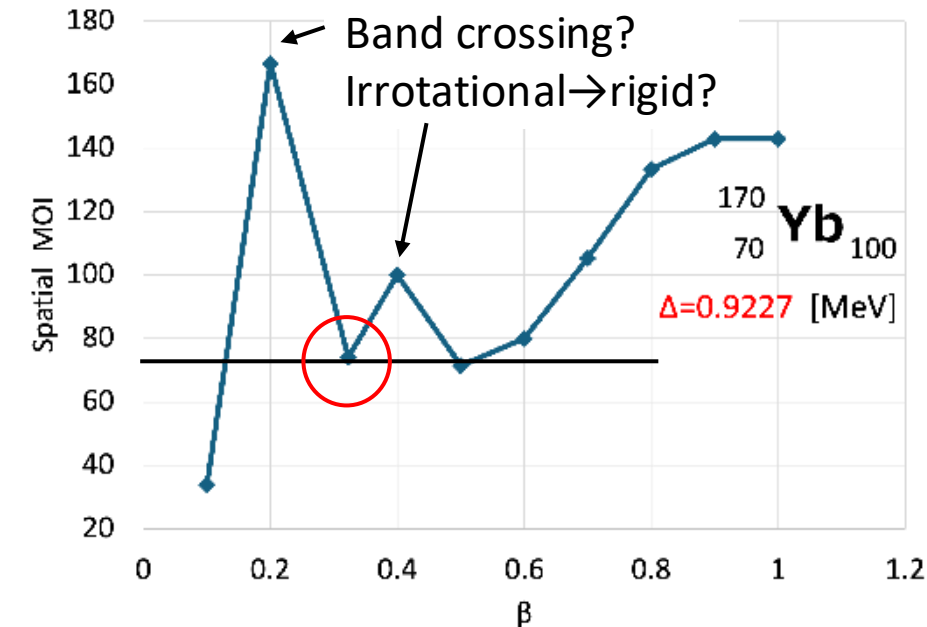


Fig. 5 : β dependence of angular velocity ω (left) and β dependence of spatial MOI (right) ($\Delta \sim 0$ [MeV])



$^{170}_{70}\text{Yb}_{100}$
 Experimental MOI: $I \sim 70$ 1/MeV
 $\beta = 0.322$
 $\Delta = 0.922$ MeV

The MOI in spatial rotation roughly increases with β , different from pair rotation.

Deformation in Sn isotopes

results 28

H. Nakada et al. Phys. Rev. C, 108, 054308, (2023)

“ $^{106-110}\text{Sn}$ are soft against the quadrupole deformation”

Calculated by HFB+QRPA (interaction type: M3Y-P6)

A. P. Zuker et al. Phys. Rev. C, 103, 024322, (2021)

“ $N \leq 64$ Sn isotopes have weak permanent quadrupole deformation”

Calculated by pseudo-SU3

So, we calculate deformed Sn isotopes

We used HFB code

Version : **HFBTHO** v2.00d

Interaction type : **SLy4**

Pairing type : **mixed**

We set the quadrupole moment $Q = -3[b] - +3[b]$ ($\beta \sim -0.2-0.2$)

$[b] = [100\text{fm}^2]$

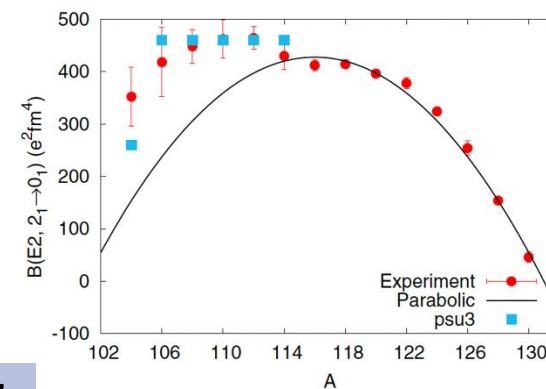


FIG. 1. The experimental $B(E2 : 2_1^+ \rightarrow 0_1^+)$ for the Sn isotopes from compilations [1], compared with some arbitrary parabolic shape and pseudo-SU3 results to be explained here (squares).

A.P. Zuker et al. Phys. Rev. .C, 103, 024322, (2021)

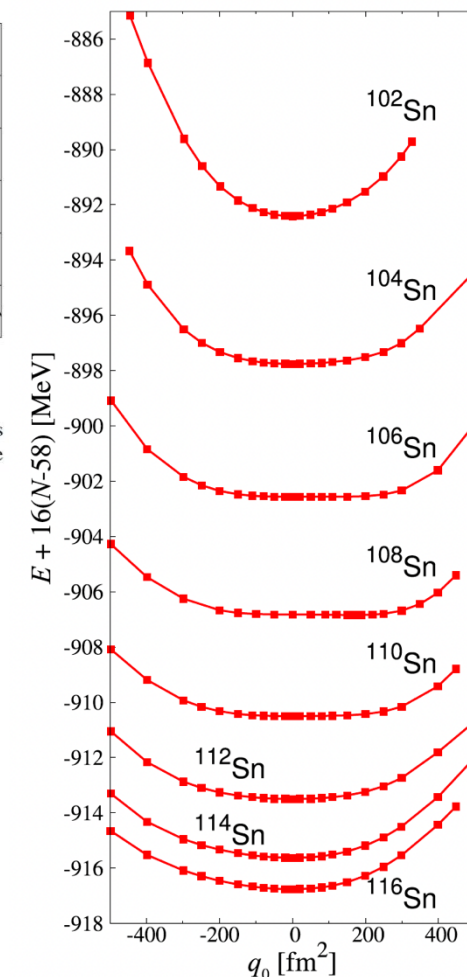
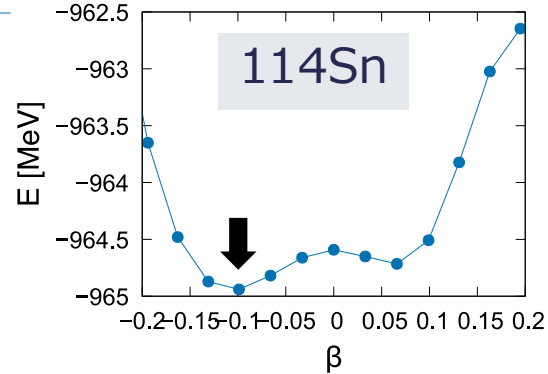
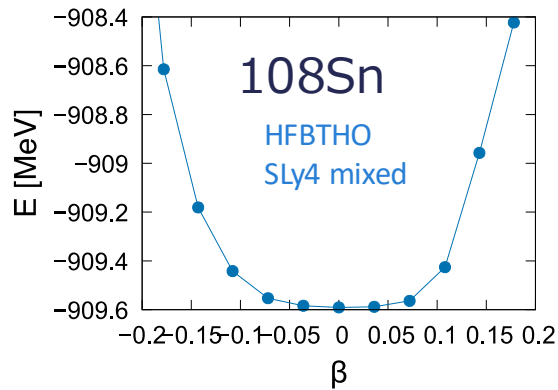


FIG. 5. Potential energy curves for $^{102-116}\text{Sn}$ obtained from the axial CHFB calculations with M3Y-P6.

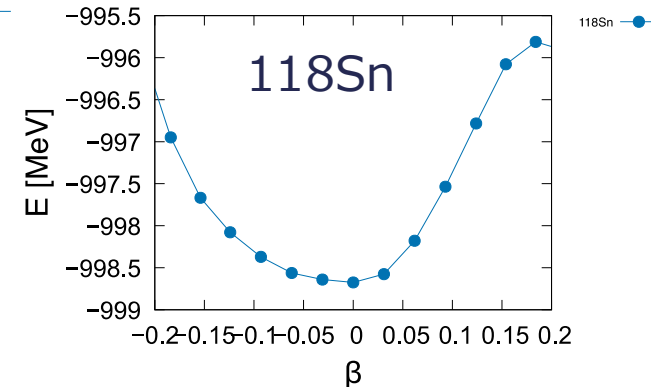
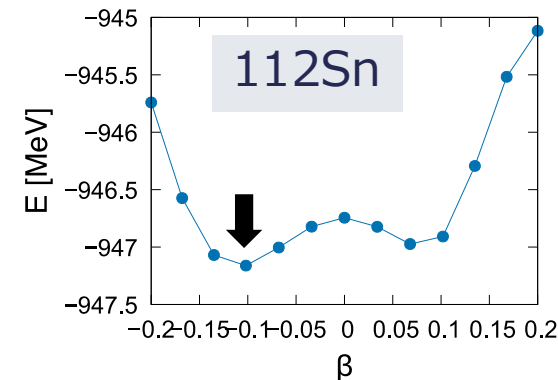
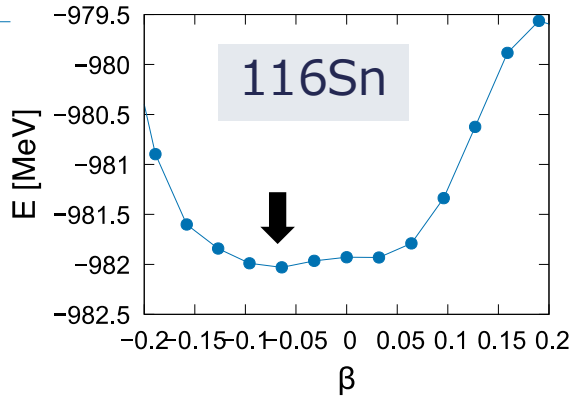
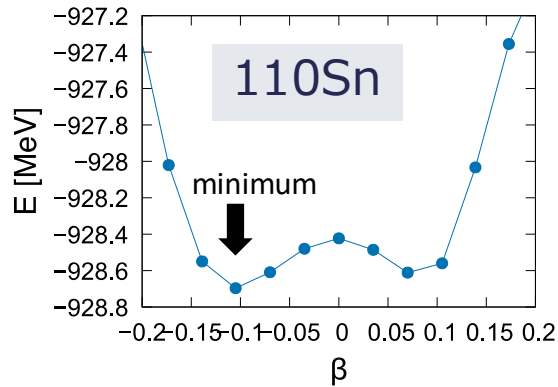
H. Nakada et al. Phys. Rev. C, 108, 054308, (2023)

Deformation in Sn isotopes



We set the quadrupole moment $Q = -3[b] - +3[b]$ ($\beta \sim -0.2-0.2$)

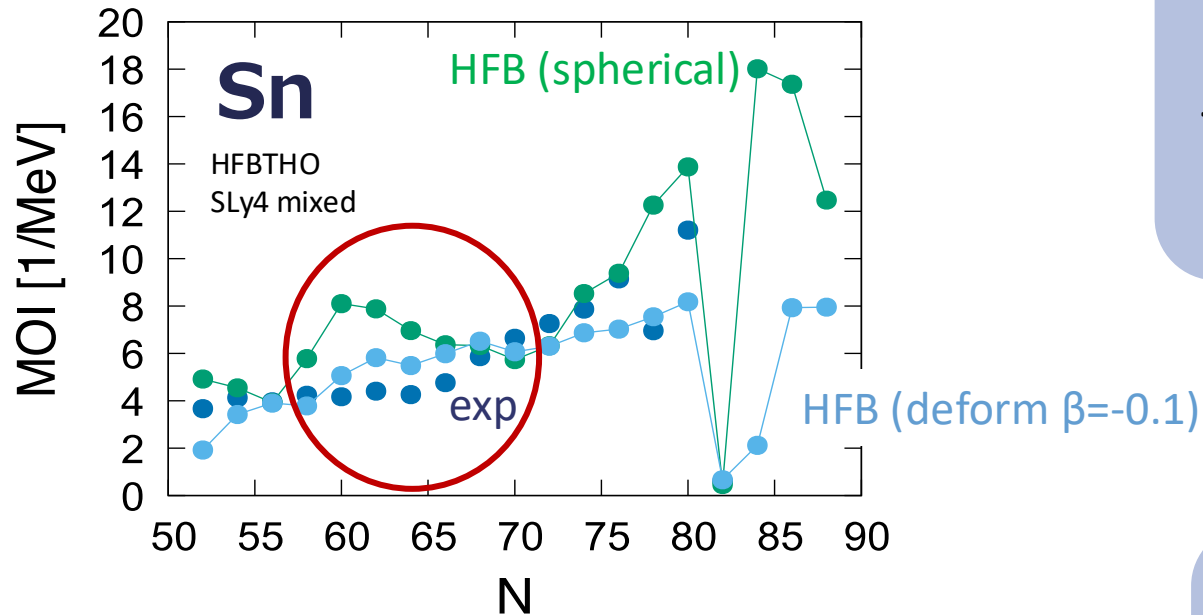
$$Q = \sqrt{\frac{5}{\pi}} \langle r^2 \rangle \beta$$



110–116Sn have weak quadrupole deformation

(Average $\beta = -0.1$, $Q = -3[b]$)

So we fix the quadrupole moment $Q=-3[b]$ ($\beta \sim -0.1$)



The deformed ($\beta=-0.1$) calculation is more fitted to experimental MOI in $^{108-116}\text{Sn}$ region

We can find ...

The first half of Sn isotopes is weakly deformed by calculating MOI from the ground state energy

$$\text{MOI}(N) = \frac{1}{\frac{\partial^2 H}{\partial N^2}} = \frac{4}{E_0(N-2) - 2E_0(N) + E_0(N+2)}$$

Pairing term in Skyrme HFB

$$\tilde{h}_q(\mathbf{r}, \sigma, \sigma') = V_0 \left(1 - V_1 \left(\frac{\rho}{\rho_0} \right)^\gamma \right) \tilde{\rho}_q$$

Pairing strength (neutron, V0) : -267.9 MeV

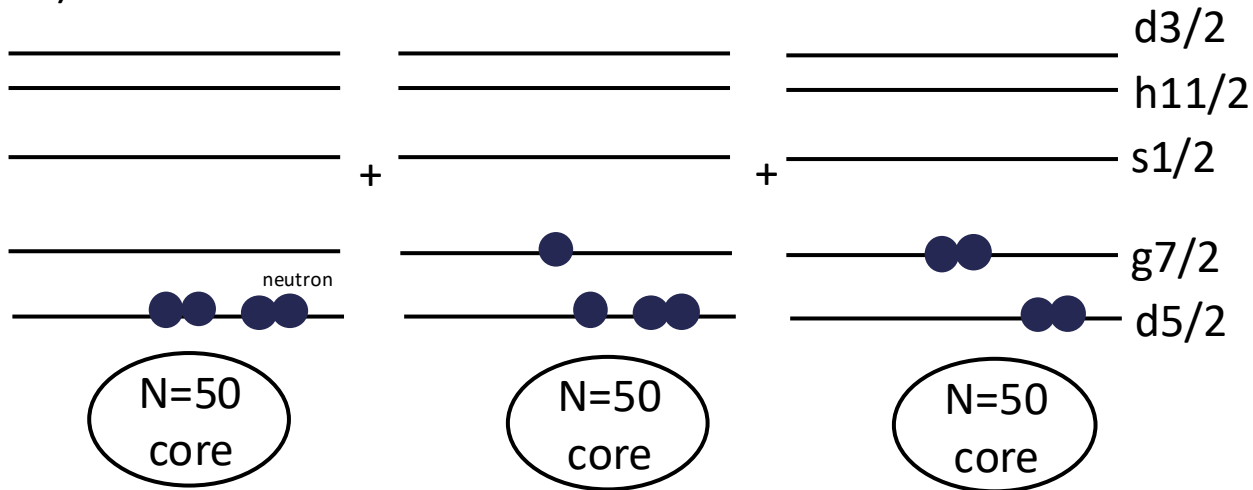
We did the shell-model calculation
to compare with the mean-field calculation

We use **KSHELL** code

N. Shimizu et al. Computer Physics Communications 244 (2019) 372–384

$$\hat{H} = \sum_i \epsilon_i a_i^\dagger a_i + \sum_{ijkl} V_{ijkl} a_i^\dagger a_j^\dagger a_l a_k$$

Ex) 104Sn



$$|\Phi\rangle = v_1|m_1\rangle + v_2|m_2\rangle + v_3|m_3\rangle + \dots$$

From solving this eigenvalue equation, we obtain the energy of nuclei E

$$\hat{H}|\Phi\rangle = E|\Phi\rangle$$

We use simple pairing interaction

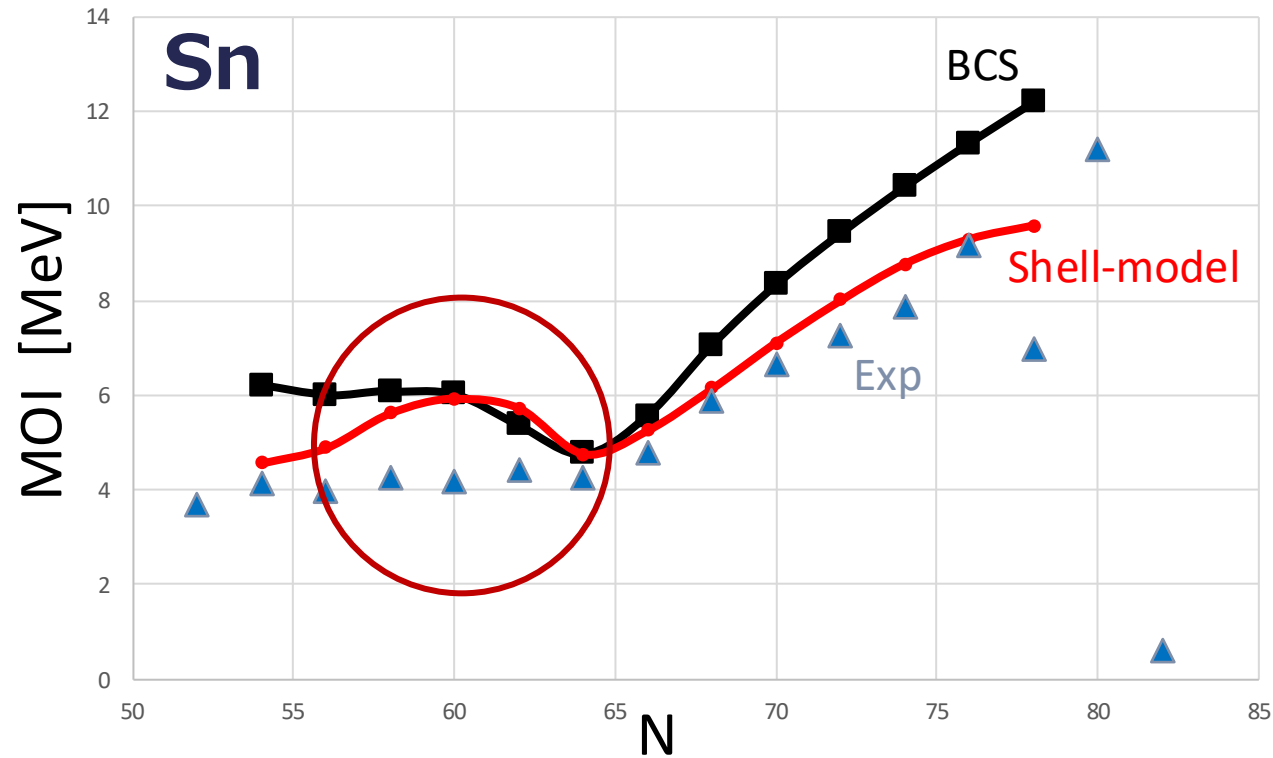
$$\sum_{mn} V_{m\bar{m}n\bar{n}}^{J=0} a_m^\dagger a_{\bar{m}}^\dagger a_n a_{\bar{n}} \quad \bar{m}, \bar{n} : \text{time reversal}$$

$$V_{m\bar{m}n\bar{n}}^{J=0} = -G(N) \sqrt{(2j_m + 1)(2j_n + 1)}$$

j_m : total angular momentum

$$G(N) = G(66) \times (N/66)$$

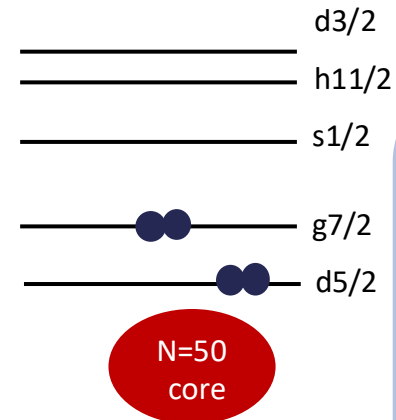
Same as the BCS setting!



$$MOI(N) = \frac{1}{\frac{\partial^2 H}{\partial N^2}} = \frac{4}{E_0(N-2) - 2E_0(N) + E_0(N+2)}$$

※ Total energy of shell-model calculation $E(N)$ is only 1-2MeV less than that of BCS

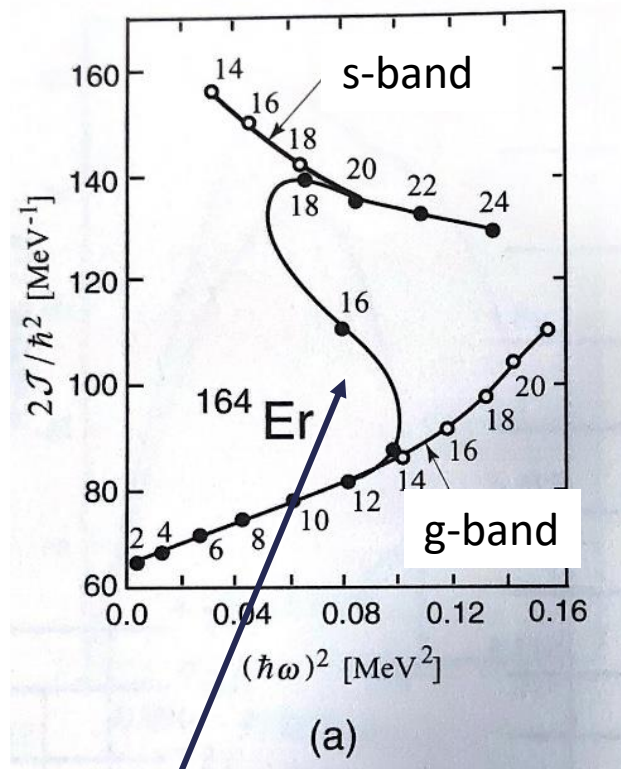
- The pair MOI with Shell-model calculation fits well to the experimental value after $N=64$
- In $^{106-114}\text{Sn}$ region, Shell-model calculation does not match to the experimental value



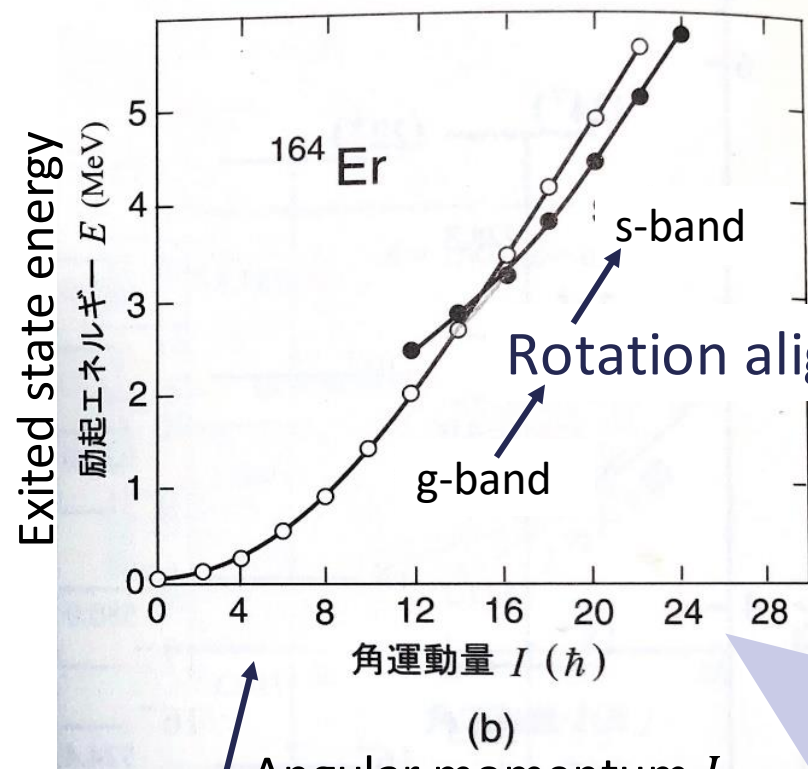
We may need to **break the core of ^{100}Sn** and allow to excite from under $N=50$?

Supplement : Back-bending

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Back-bending



Band crossing

Back-bending :

MOI suddenly increases at angular momentum $I=14,16$ with respect to angular velocity ω

because

Band-crossing :

Rotational energy jump from g-band to s-band

Whether there is a similar phenomenon in pairing rotation? (N-MOI)

